

ROMANIAN MATHEMATICAL MAGAZINE

Find :

$$\Delta = \int_0^{\infty} \frac{\sqrt{x} \ln(x)}{1 + e^{2\pi\sqrt{x}}} dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Yang Silva Cartolin-Peru

$$I = \int_0^{\infty} \frac{\sqrt{x} \ln(x)}{1 + e^{2\pi\sqrt{x}}} dx = \int_0^{\infty} \frac{x^{\frac{1}{2}} e^{-2\pi x^{\frac{1}{2}}} \ln(x)}{1 + e^{-2\pi x^{\frac{1}{2}}}} dx$$

$$\text{We do : } x = t^2 \rightarrow dx = 2t dt$$

$$I = 4 \int_0^{\infty} \frac{t^2 e^{-2\pi t} \ln(t)}{1 + e^{-2\pi t}} dt = 4 \sum_{n=1}^{\infty} (-1)^{n-1} \int_0^{\infty} t^2 e^{-2\pi n t} \ln(t) dt$$

$$\text{We do : } x = 2\pi n t \rightarrow dt = \frac{dx}{2\pi n}$$

$$I = \frac{1}{2\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} \int_0^{\infty} x^2 e^{-x} \ln\left(\frac{x}{2\pi n}\right) dx =$$

$$\frac{1}{2\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} \left[\int_0^{\infty} x^2 e^{-x} \ln(x) dx - (\ln(2\pi) + \ln(n)) \int_0^{\infty} x^2 e^{-x} dx \right]$$

$$\text{We know that : } \Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x} dx \rightarrow \Gamma'(z) = \int_0^{\infty} x^{z-1} e^{-x} \ln(x) dx$$

$$I = \frac{1}{2\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} [\Gamma'(3) - (\ln(2\pi) + \ln(n)) \Gamma(3)] =$$

$$\frac{\Gamma'(3)}{2\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} - \frac{\ln(2\pi)}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} - \frac{1}{n^3} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} \ln(n) =$$

$$\frac{1}{\pi^3} \left(\frac{3}{2} - \gamma \right) \eta(3) - \frac{\ln(2\pi)}{\pi^3} \eta(3) - \frac{1}{\pi^3} (-\eta'(3)) =$$

$$\frac{3\zeta(3)}{4\pi^3} \left(\frac{3}{2} - \gamma \right) - \frac{3\ln(2\pi)\zeta(3)}{4\pi^3} - \frac{1}{4\pi^3} (-3\zeta'(3) - \zeta(3) \ln(2)) =$$

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$$\frac{3\zeta(3)}{4\pi^3} \left(\frac{3}{2} - \gamma - \ln(2\pi) + \frac{\zeta'(3)}{\zeta(3)} + \frac{1}{3} \ln(2) \right) = \frac{3\zeta(3)}{4\pi^3} \left(\ln \left(\frac{2^{-\frac{2}{3}} e^{\frac{3}{2}}}{\pi} \right) + \frac{\zeta'(3)}{\zeta(3)} - \gamma \right)$$

Solution 2 by Omkimar Rao-India

$$\Delta = \int_0^{\infty} \frac{\sqrt{x} \ln(x)}{1 + e^{2\pi\sqrt{x}}} dx \stackrel{x \rightarrow x^2}{=} 2 \int_0^{\infty} \frac{x^2 \ln(x)}{1 + e^{2\pi x}} dx \stackrel{2\pi x \rightarrow x}{=} \underbrace{\frac{1}{4\pi^3} \int_0^{\infty} \frac{x^2 \ln(x)}{1 + e^x} dx}_{P'(3)} - \frac{\ln(2\pi)}{4\pi^3} \int_0^{\infty} \frac{1}{1 + e^x} dx$$

$$P(k) = \int_0^{\infty} \frac{x^{k-1}}{1 + e^x} dx = \Gamma(k) \eta(k)$$

$$P'(k) = \left(\Gamma(k) (1 - 2^{1-k}) \zeta(k) \right)' =$$

$$\frac{\Gamma'(k)}{\Gamma(k)\psi(k)} (1 - 2^{1-k}) \zeta(k) + \Gamma(k) \zeta(k) (2^{1-k} \ln 2) + \Gamma(k) (1 - 2^{1-k}) \zeta'(k)$$

$$P'(3) = \underbrace{(\Gamma(3))}_2 \underbrace{(\psi(3))}_{\frac{3}{2}-\gamma} \left(\frac{3}{4} \zeta(3) \right) + \underbrace{(\Gamma(3))}_2 (\zeta(3)) \left(\frac{1}{4} \ln 2 \right) + \underbrace{(\Gamma(3))}_2 \left(\frac{3}{4} \zeta'(3) \right)$$

$$= \frac{1}{4\pi^3} \left((3 - 2\gamma) \left(\frac{3}{4} \zeta(3) \right) + \zeta(3) \left(\frac{1}{2} \ln 2 \right) + \left(\frac{3}{2} \zeta'(3) \right) \right) - \frac{\ln(2\pi)}{4\pi^3} (\Gamma(1) \eta(1))$$

$$\therefore \int_0^{\infty} \frac{\sqrt{x} \ln(x)}{1 + e^{2\pi\sqrt{x}}} dx = \frac{1}{4\pi^3} \left((3 - 2\gamma) \left(\frac{3}{4} \zeta(3) \right) + \zeta(3) \left(\frac{1}{2} \ln 2 \right) + \left(\frac{3}{2} \zeta'(3) \right) \right) - \frac{(\ln(2\pi))(\ln 2)}{4\pi^3}$$

Solution 3 by Lara Gangler-Hungary

$$\Delta = \int_0^{\infty} \frac{\sqrt{x} \ln(x)}{1 + e^{2\pi\sqrt{x}}} dx \quad \therefore y = 2\pi\sqrt{x} \rightarrow x = \frac{y^2}{4\pi^2} \rightarrow dx = \frac{y}{2\pi^2} dy$$

$$I = \int_0^{\infty} \frac{\frac{y}{2\pi} \ln \frac{y^2}{4\pi^2}}{1 + e^y} \frac{y}{2\pi^2} dy = \frac{1}{2\pi^3} \int_0^{\infty} \frac{y^2 [\ln(y) - \ln(2\pi)]}{1 + e^y} dy = \frac{1}{2\pi^3} (I_1 - I_2)$$

As $0 \leq e^{-y} \leq 1$, the integrand within both integrals can be extended to a power series:

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$$\begin{aligned}
 I_1 &= \int_0^{\infty} \frac{y^2 \ln(y)}{1+e^y} dy = \int_0^{\infty} \frac{y^2 \ln(y) e^{-y}}{1+e^{-y}} dy = \int_0^{\infty} y^2 \ln(y) e^{-y} \sum_{k=0}^{\infty} e^{-ky} (-1)^k dy = \\
 &= \int_0^{\infty} \sum_{k=0}^{\infty} y^2 \ln(y) e^{-(k+1)y} (-1)^k dy = \sum_{k=0}^{\infty} \int_0^{\infty} y^2 \ln(y) e^{-(k+1)y} (-1)^k dy
 \end{aligned}$$

Note 1*: gamma function and its derivative

$$\int_0^{\infty} y^{a-1} e^{-by} \ln(y) dy = \Gamma(a) \frac{\Psi(a) - \ln(b)}{b^a}$$

$$I_1 = \sum_{k=0}^{\infty} \int_0^{\infty} y^2 \ln(y) e^{-(k+1)y} (-1)^k dy = \sum_{k=0}^{\infty} \Gamma(3) \frac{\Psi(3) - \ln(k+1)}{(k+1)^3} (-1)^k =$$

$$\sum_{k=1}^{\infty} 2 \frac{\left(\frac{1}{2} + 1 + \Psi(1)\right) - \ln(k)}{k^3} (-1)^{k-1} = 2(I_{11} - I_{12})$$

$$\begin{aligned}
 I_{11} &= \sum_{k=1}^{\infty} \int_0^{\infty} \frac{1 + \frac{1}{2} + \Psi(1)}{k^3} (-1)^{k-1} = \sum_{k=1}^{\infty} \frac{\frac{3}{2} - \gamma}{k^3} (-1)^{k-1} = \left(\frac{3}{2} - \gamma\right) \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3} \\
 &= \left(\frac{3}{2} - \gamma\right) \eta(3) = \left(\frac{3}{2} - \gamma\right) \left(1 - \frac{1}{2^{1-3}}\right) \zeta(3) = \frac{9-6\gamma}{8} \zeta(3)
 \end{aligned}$$

where γ is the Euler – Mascheroni constant: $\Psi(1) = -\gamma$

$$\begin{aligned}
 &= \int_0^{\infty} e^{-y} \ln(y) dy \text{ and } \eta \text{ is the Dirichlet eta function: } \eta(a) \\
 &= (1 - 2^{1-a}) \zeta(a)
 \end{aligned}$$

Note 2*: derivative of the Dirichlet eta function

$$\sum_{k=1}^{\infty} \frac{\ln(k)}{k^a} (-1)^{k-1} = -[2^{1-a} \ln(2) \zeta(a) + (1 - 2^{1-a}) \zeta'(a)]$$

$$\begin{aligned}
 \rightarrow I_{12} &= \sum_{k=1}^{\infty} \frac{\ln(k)}{k^3} (-1)^{k-1} = -[2^{1-2} \ln(2) \zeta(3) + (1 - 2^{1-2}) \zeta'(3)] \\
 &= -\frac{\ln(2)}{4} \zeta(3) - \frac{3}{4} \zeta'(3)
 \end{aligned}$$

$$I_1 = 2(I_{11} - I_{12}) = 2 \left(\frac{9-6\gamma}{8} \zeta(3) + \frac{\ln(2)}{4} \zeta(3) + \frac{3}{4} \zeta'(3) \right) = \frac{9+2\ln(2)-6\gamma}{4} \zeta(3) + \frac{3}{2} \zeta'(3)$$

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$$\begin{aligned}
 I_2 &= \ln(2\pi) \int_0^\infty \frac{y^2}{1+e^y} dy = \ln(2\pi) \int_0^\infty y^2 e^{-y} \sum_{k=0}^\infty e^{-ky} (-1)^k dy = \\
 &= \ln(2\pi) \sum_{k=0}^\infty \int_0^\infty y^2 e^{-(k+1)y} (-1)^k dy = \ln(2\pi) \sum_{k=0}^\infty \Gamma(3) \frac{(-1)^k}{(k+1)^3} = 2\eta(3) \ln(2\pi) \\
 &= \frac{3}{2} \zeta(3) \ln(2\pi)
 \end{aligned}$$

$$I = \frac{1}{2\pi^3} (I_1 - I_2) = \frac{1}{2\pi^3} \left(\frac{9 + 2 \ln(2) - 6\gamma}{4} \zeta(3) + \frac{3}{2} \zeta'(3) - \frac{3}{2} \zeta(3) \ln(2\pi) \right)$$

$$I = \frac{1}{8\pi^3} \left[(9 - \ln(16\pi^6) - 6\gamma) \zeta(3) + \frac{6}{4} \zeta'(3) \right]$$

Note 1* Statement: $\int_0^\infty y^{a-1} e^{-by} \ln(y) dy = \Gamma(a) \frac{\Psi(a) - \ln(b)}{b^a}$

Proof: Differentiating the left side with respect to a :

$$\frac{\partial}{\partial a} \int_0^\infty y^{a-1} e^{-by} dy = \frac{\partial}{\partial(a-1)} \int_0^\infty y^{a-1} e^{-by} dy = \int_0^\infty y^{a-1} e^{-by} \ln(y) dy$$

Differentiating the right side with respect to a :

$$\frac{\partial \frac{\Gamma(a)}{b^a}}{\partial a} = \frac{\Gamma(a)\Psi(a)b^a - \Gamma(a)b^a \ln(b)}{b^{2a}} = \Gamma(a) \frac{\Psi(a) - \ln(b)}{b^a}$$

where Ψ is the digamma function, the logarithmic derivative of the gamma function:

$$\Psi(a) = \frac{d \ln \Gamma(a)}{d a} = \frac{\frac{d \Gamma(a)}{d a}}{\Gamma(a)} = \frac{\Gamma'(a)}{\Gamma(a)}$$

$$\rightarrow \int_0^\infty y^{a-1} e^{-by} \ln(y) dy = \Gamma(a) \frac{\Psi(a) - \ln(b)}{b^a}, QED$$

Note 2* Statement: $\sum_{k=1}^\infty \frac{\ln(k)}{k^a} (-1)^{k-1} = -[2^{1-a} \ln(2) \zeta(a) + (1 - 2^{1-a}) \zeta'(a)]$

Proof:

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$$\begin{aligned}
 \sum_{k=1}^{\infty} \frac{\ln(k)}{k^a} (-1)^{k-1} &= \sum_{k=1}^{\infty} k^{-a} (-1)^{k-1} \ln(k) = - \sum_{k=1}^{\infty} (-1)^{k-1} k^{-a} [-\ln(k)] \\
 &= - \sum_{k=1}^{\infty} (-1)^{k-1} \frac{dk^{-a}}{da} = - \frac{d}{da} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^a} = - \frac{d\eta(a)}{da} \\
 &= - \frac{d}{da} [(1 - 2^{1-a})\zeta(a)] = -[2^{1-a} \ln(2) \zeta(a) + (1 - 2^{1-a})\zeta'(a)], \quad QED
 \end{aligned}$$

Solution 4 by Quadri Faruk Temitope-Nigeria

$$\Delta = \int_0^{\infty} \frac{\sqrt{x} \ln(x)}{1 + e^{2\pi\sqrt{x}}} dx = \int_0^{\infty} \frac{x \ln(x^2)}{1 + e^{2\pi x}} \cdot 2x dx = 4 \int_0^{\infty} \frac{x^2 \ln(x)}{1 + e^{2\pi x}} dx =$$

$$\text{Recall that : } \frac{1}{1 + e^{2\pi x}} = \sum_{n=1}^{\infty} (-1)^{n-1} e^{-2\pi n x}$$

$$= 4 \sum_{n=1}^{\infty} (-1)^{n-1} \int_0^{\infty} x^2 e^{-2\pi n x} \ln(x) dx = 4 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{d}{ds} \int_0^1 x^{s-1} e^{-2\pi n x} dx =$$

$$I(s) = 4 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{d}{ds} \int_0^{\infty} x^{s-1} e^{-2\pi n x} dx = \begin{cases} 2\pi n = y, & x = \frac{y}{2\pi n} \\ dx = \frac{dy}{2\pi n} \end{cases}$$

$$I(s) = 4 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{d}{ds} \int_0^{\infty} \left(\frac{y}{2\pi n}\right)^{s-1} e^{-y} \frac{dy}{2\pi n} = 4 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{d}{ds} \left(\frac{1}{2\pi n}\right)^s \underbrace{\int_0^{\infty} y^{s-1} e^{-y} dy}_{\Gamma(s)} =$$

This :

$$4 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{d}{ds} \left(\frac{1}{2\pi n}\right)^s \Gamma(s) = 4 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{d}{ds} \frac{\Gamma(s)}{(2\pi n)^s} = -4 \frac{d}{ds} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} \frac{\Gamma(s)}{(2\pi)^s} =$$

$$4 \frac{d}{ds} \frac{\Gamma(s)}{(2\pi)^s} \eta(s)$$

$$\therefore \eta(s) = (1 - 2^{1-s})\zeta(s)$$

$$I(s) = 4 \frac{d}{ds} \frac{\Gamma(s)(1 - 2^{1-s})\zeta(s)}{(2\pi)^s}$$

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$$I(s) = 4 \frac{\Gamma(s)\eta(s)}{(2\pi)^s} \left[\psi(s) - \ln(2\pi) + \frac{\eta'(s)}{\eta(s)} \right]$$

$$s = 3$$

$$I = 4 \frac{\Gamma(3)\eta(3)}{(2\pi)^3} \left[\psi(3) - \ln(2\pi) + \frac{\eta'(3)}{\eta(3)} \right]$$

$$I = \frac{3\zeta(3)}{4\pi^3} \left(\frac{3}{2} - \gamma - \ln(2\pi) + \frac{\zeta'(3)}{\zeta(3)} + \frac{1}{3} \ln(2) \right) = \frac{3\zeta(3)}{4\pi^3} \left(\ln \left(\frac{2^{-\frac{2}{3}} e^{\frac{3}{2}}}{\pi} \right) + \frac{\zeta'(3)}{\zeta(3)} - \gamma \right)$$