ROMANIAN MATHEMATICAL MAGAZINE

Find :

$$\Delta = \int_0^\infty \frac{x + \sqrt[3]{x^2}}{1 + e^{2x}} dx$$

Proposed by Shirvan Tahirov-Azerbaijan

Solution 1 by Yang Silva Cartolin-Peru

$$I = \int_0^\infty \frac{x + \sqrt[3]{x^2}}{1 + e^{2x}} dx = \int_0^\infty \frac{e^{-2x}\left(x + x^{\frac{2}{3}}\right)}{1 + e^{-2x}} dx = \sum_{n=0}^{\infty} (-1)^n \int_0^\infty e^{-2(n+1)x}\left(x + x^{\frac{2}{3}}\right) dx$$

$$I = -\sum_{n=0}^{\infty} (-1)^n \int_0^\infty e^{-2nx}\left(x + x^{\frac{2}{3}}\right) dx \,.\, Let: S(k) = \sum_{n=1}^{\infty} (-1)^n \int_0^\infty e^{-2nx} x^k dx$$

$$We\ \ do : 2nx = t \to dx = \frac{dt}{2n} \to S(k) = \frac{1}{2^{k+1}} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{k+1}} \int_0^\infty t^k e^{-t} dt$$

$$S(k) = -\frac{\Gamma(k+1)}{2^{k+1}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{k+1}} = -\frac{\Gamma(k+1)}{2^{k+1}} \eta(k+1) = -\frac{\Gamma(k+1)}{2^{k+1}} (1 - 2^{-k})\zeta(k+1)$$

$$Therefore:$$

$$I = -\left(S(1) + S\left(\frac{2}{3}\right)\right) = \frac{\Gamma(2)}{2^3}\zeta(2) + \frac{\Gamma\left(\frac{5}{3}\right)}{2^{\frac{5}{3}}}(1 - 2^{-\frac{2}{3}})\zeta\left(\frac{5}{3}\right)$$

$$I = \frac{\pi^2}{48} + \frac{\Gamma\left(\frac{5}{3}\right)}{\sqrt[3]{2^5}}\zeta\left(\frac{5}{3}\right) - \frac{\Gamma\left(\frac{5}{3}\right)}{\sqrt[3]{2^7}}\zeta\left(\frac{5}{3}\right), \qquad I = \frac{\pi^2}{48} + \Gamma\left(\frac{5}{3}\right)\zeta\left(\frac{5}{3}\right)\left(\frac{2\sqrt[3]{2} - \sqrt[3]{4}}{8}\right)$$

Solution 2 by Samuel Ajibade-Nigeria

$$\Delta = \int_0^\infty \frac{x + \sqrt[3]{x^2}}{1 + e^{2x}} dx = \int_0^\infty \frac{x + x^{\frac{2}{3}}}{1 + e^{2x}} dx = \int_0^\infty \frac{x + x^{\frac{2}{3}}}{1 - (-e^{-2x})} e^{-2x} dx$$

$$\Delta = \int_0^\infty \sum_{n=0}^{\infty} (-1)^n e^{-2x(n+1)} \left(x + x^{\frac{2}{3}}\right) dx \quad \{geometric\ \ series:\ |e^{-2x}| < 1\}$$

$$\Delta = \int_0^\infty \sum_{k=1}^{\infty} (-1)^{k-1} e^{-2kx} \left(x + x^{\frac{2}{3}}\right) dx \quad \{n+1=k\}$$

$$\Delta = \sum_{k=1}^{\infty} (-1)^{k-1} \int_0^\infty \left(x + x^{\frac{2}{3}}\right) e^{-2kx} dx \quad \{Fubini - Tonelli\ \ Theorem\}$$

$$\Delta = \sum_{k=1}^{\infty} (-1)^{k-1} \left(\frac{1}{(2k)^2} + \frac{\frac{2}{3}\Gamma\left(\frac{2}{3}\right)}{(2k)^{\frac{2}{3}+1}} \right) \{Laplace\ \ Transforms\}$$

$$\Delta = \frac{1}{4} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k^2} + \frac{2^{-\frac{2}{3}}}{3} \Gamma\left(\frac{2}{3}\right) \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k^{\frac{5}{3}}}$$

$$\Delta = \frac{1}{4}\eta(2) + \frac{2^{-\frac{2}{3}}}{3} \frac{2\pi}{\sqrt{3}\Gamma\left(\frac{1}{3}\right)} \eta\left(\frac{5}{3}\right), \qquad \Delta = \frac{\pi^2}{48} + \Gamma\left(\frac{5}{3}\right) \zeta\left(\frac{5}{3}\right) \left(\frac{2\sqrt[3]{2} - \sqrt[3]{4}}{8} \right)$$

Solution 3 by Quadri Faruk Temitope-Nigeria

$$\Delta = \int_0^\infty \frac{x + \sqrt[3]{x^2}}{1 + e^{2x}} dx = \sum_{n=1}^{\infty} (-1)^{n-1} \int_0^\infty (x + \sqrt[3]{x^2}) e^{-2nx} dx =$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \int_0^\infty \left[\frac{y}{2n} + \left(\frac{y}{2n}\right)^{\frac{2}{3}} \right] e^{-y} \frac{dy}{2n} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{2n} \int_0^\infty \left[\frac{y}{2n} + \left(\frac{y}{2n}\right)^{\frac{2}{3}} \right] e^{-y} dy =$$

$$-\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)^2} \int_0^\infty y e^{-y} dy - \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)^{\frac{5}{3}}} \int_0^\infty y^{\frac{2}{3}} e^{-y} dy =$$

$$-\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)^2} \Gamma(2) - \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)^{\frac{5}{3}}} \Gamma\left(\frac{5}{3}\right) = -\frac{1}{4}\left(-\frac{\pi^2}{12}\right) + \frac{1}{2^{\frac{5}{3}}} \Gamma\left(\frac{5}{3}\right) \eta\left(\frac{5}{3}\right) =$$

$$\frac{\pi^2}{48} + \frac{1}{2^{\frac{5}{3}}} \left(1 - 2^{1-\frac{5}{3}}\right) \zeta\left(\frac{5}{3}\right) = \frac{\pi^2}{48} + \frac{\left(2^{\frac{2}{3}} - 1\right)}{2^{\frac{7}{3}}} \zeta\left(\frac{5}{3}\right)$$

$$\Delta = \frac{\pi^2}{48} + \Gamma\left(\frac{5}{3}\right) \zeta\left(\frac{5}{3}\right) \left(\frac{2\sqrt[3]{2} - \sqrt[3]{4}}{8} \right)$$

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Solution 4 by Omkumar Rao-India

$$\int_0^\infty \frac{x^{k-1}}{e^x+1}dx = \Gamma(k)\eta(k)$$

$$I = \underbrace{\int_0^\infty \frac{x}{1+e^{2x}}dx}_{2x\to x} + \underbrace{\int_0^\infty \frac{x^{\frac{2}{3}}}{1+e^{2x}}dx}_{2x\to x} = \frac{1}{4}\underbrace{\int_0^\infty \frac{x}{1+e^x}dx}_{\Gamma(2)\eta(2)} + \frac{1}{2^{\frac{5}{3}}}\underbrace{\int_0^\infty \frac{x^{\frac{2}{3}}}{1+e^x}dx}_{\Gamma\left(\frac{5}{3}\right)\eta\left(\frac{5}{3}\right)}$$

$$I = \frac{1}{4}\left(\frac{\pi^2}{12}\right) + \frac{1}{2^{\frac{5}{3}}}\left(\frac{2}{3}\Gamma\left(\frac{2}{3}\right)\eta\left(\frac{5}{3}\right)\right)$$

$$\boxed{\int_0^\infty \frac{x+\sqrt[3]{x^2}}{1+e^{2x}}dx = \frac{\pi^2}{48} + \frac{\Gamma\left(\frac{2}{3}\right)\eta\left(\frac{5}{3}\right)}{3.2^{\frac{2}{3}}}}$$