ROMANIAN MATHEMATICAL MAGAZINE

Find **:**

$$\Delta=\int_0^1 x\ln\big(x\sin(x)\big)dx$$

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Solution 1 by Quadri Faruk Temitope-Nigeria

$$\therefore \textit{Recall that}: \ln(AB)=\ln(A)+\ln(B)$$

$$\Delta=\int_0^1 x\ln\big(x\sin(x)\big)dx=\int_0^1 x\ln(x)dx+\int_0^1 x\ln\big(\sin(x)\big)dx=$$

$$\therefore \textit{Recall that}: \ln(\sin(x))=-\ln(2)-\sum_{n=1}^{\infty}\frac{\cos(2nx)}{n}$$

$$\Delta=\underbrace{\int_0^1 x\ln(x)dx}_{I.B.P}-\ln(2)\int_0^1 xdx-\sum_{n=1}^{\infty}\frac{1}{n}\underbrace{\int_0^1 x\cos(2nx)\,dx}_{I.B.P}=$$

$$=\frac{x^2}{2}\ln(2)\bigg|_0^1-\int_0^1\frac{x^2}{2}.\frac{1}{x}dx-\frac{x^2}{2}\ln(2)\bigg|_0^1-$$

$$\sum_{n=1}^{\infty}\frac{1}{n}\left[\frac{\sin(2n)}{n}+\frac{\cos(2n)}{4n^2}-\frac{1}{4n^2}\right]=-\frac{1}{2}\int_0^1 x\,dx-\frac{\ln(2)}{2}-$$

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{\sin(2n)}{n^2}-\frac{1}{4}\sum_{n=1}^{\infty}\frac{\cos(2n)}{n^3}+\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^3}=-\frac{1}{4}-\frac{\ln(2)}{2}-$$

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{\sin(2n)}{n^2}-\frac{1}{4}\sum_{n=1}^{\infty}\frac{\cos(2n)}{n^3}+\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^3}=$$

$$-\frac{1}{4}-\frac{\ln(2)}{2}-\frac{1}{2}Cl_2(2)-\frac{1}{4}Cl_3(2)+\frac{1}{4}\zeta(3)$$

$$\Delta=\int_0^1 x\ln\big(x\sin(x)\big)dx$$

$$\Delta=-\frac{1}{4}-\frac{\ln(2)}{2}-\frac{1}{2}Cl_2(2)-\frac{1}{4}Cl_3(2)+\frac{1}{4}\zeta(3)$$

Note :

$$\therefore \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^2} = Cl_2(\theta) \quad \therefore \sum_{n=1}^{\infty} \frac{\cos(\theta n)}{n^3} = Cl_3(\theta)$$

$$Cl_n(\theta) - Clausen \ \ function$$

Solution 2 by Yang Silva Cartolin-Peru

$$We \ \ known \ that:$$

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i} \to 2ie^{ix}\sin(x) = 1 - e^{2ix}$$

$$-2ixe^{ix}\sin(x) = x(1 - e^{2ix}) \to \log(2) + i\left(x - \frac{\pi}{2}\right) + \log\big(x\sin(x)\big) =$$

$$\log(x) + \log(1 - e^{2ix})$$

$$\Re\{\log\big(x\sin(x)\big)\} = \log(x) + \Re\{\log(1 - e^{2ix})\} - \log(2)\int_0^1 xdx$$

$$There\ fore: I = \int_0^1 x\log(x)\,dx \ + \Re\left\{\int_0^1 \mathrm{x}\log(1 - e^{2ix})\right\}dx - \log(2)\int_0^1 xdx$$

$$I = \xi_1 + \xi_2 - \frac{1}{2}\log(2) - -(1)$$

$$\xi_1 = \int_0^1 x\log(x)\,dx \ \ I.B.P: \begin{cases} u = \log(x), & du = \dfrac{dx}{x} \\ dv = xdx, & v = \dfrac{x^2}{2} \end{cases}$$

$$\xi_1 = \left[\log(x)\frac{x^2}{2}\right]_0^1 - \frac{1}{2}\int_0^1 xdx = -\frac{1}{4}$$

$$\xi_2 = \Re\left\{\int_0^1 \mathrm{x}\log(1 - e^{2ix})\right\} = -\Re\left\{\sum_{n=1}^{\infty}\frac{1}{n}\int_0^1 xe^{2ixn}dx\right\}$$

$$\xi_2 = -\sum_{n=1}^{\infty}\frac{1}{n}\int_0^1 x\cos(2nx)dx = -\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\sin(2n)}{2n} - \frac{\cos(2n) - 1}{4n^2}\right)$$

$$\xi_2=\frac{1}{2}\sum_{n=1}^{\infty}\frac{\sin(2n)}{n^2}-\frac{1}{4}\sum_{n=1}^{\infty}\frac{\cos(2n)}{n^3}+\frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^3},\qquad \xi_2=-\frac{1}{2}Cl_2(2)-\frac{1}{4}Cl_3(2)+\frac{1}{4}\zeta(3)$$

There fore :

$$I=-\frac{1}{4}-\frac{\ln(2)}{2}-\frac{1}{2}Cl_2(2)-\frac{1}{4}Cl_3(2)+\frac{1}{4}\zeta(3)$$

Solution 3 by Omkumar Rao-India

$$I=\underbrace{\int_0^1 x\ln(x)\,dx}_{-\frac{1}{4}}+\underbrace{\int_0^1 x\ln(\sin x)\,dx}_{\ln(\sin x)=-\ln 2-\sum_{m=1}^{\infty}\frac{\cos(2mx)}{m}}$$

$$I=-\frac{1}{4}+\int_0^1 x\left(-\ln 2-\sum_{m=1}^{\infty}\frac{\cos(2mx)}{m}\right)dx=-\frac{1}{4}-\frac{1}{2}\ln 2-\sum_{m=1}^{\infty}\frac{1}{m}\int_0^1 x\cos(2mx)\,dx$$

$$I=-\frac{1}{4}-\frac{1}{2}\ln 2-\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{\sin(2m)}{2m}-\frac{1}{4m^2}+\frac{\cos(2m)}{4m^2}\right)$$

$$I=-\frac{1}{4}-\frac{1}{2}\ln 2-\frac{1}{2}\underbrace{\sum_{m=1}^{\infty}\frac{\sin(2m)}{m^2}}_{Im\left(Li_2(e^{2i})\right)/Cl_2(2)}+\frac{1}{4}\underbrace{\sum_{m=1}^{\infty}\frac{1}{m^3}}_{\zeta(3)}-\frac{1}{4}\underbrace{\sum_{m=1}^{\infty}\frac{\cos(2m)}{m^3}}_{\Re\left(Li_3(e^{2i})\right)/Cl_3(2)}$$

$$\left(Cl_2(2)\right)=\textit{Clausen function of order } 2$$

$$\left(Cl_3(2)\right)=\textit{Clausen function of order } 3$$

$$I=-\frac{1}{4}-\frac{1}{2}\ln 2-\frac{1}{2}\left(Im\left(Li_2(e^{2i})\right)\right)+\frac{1}{4}\zeta(3)-\frac{1}{4}\Re\left(Li_3(e^{2i})\right)$$

Or

$$I=-\frac{1}{4}-\frac{1}{2}\ln 2-\frac{1}{2}\left(Cl_2(2)\right)+\frac{1}{4}\zeta(3)-\frac{1}{4}\left(Cl_3(2)\right)$$