ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\triangle=\int_0^\infty \frac{x^2\left(log(x)+arctan(x)\right)}{1+exp(\pi x)}dx$$

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$$\triangle=\int_0^\infty \frac{x^2\left(log(x)+arctan(x)\right)}{1+exp(\pi x)}dx=\int_0^\infty \frac{x^2 log(x)}{1+e^{\pi x}}dx+\int_0^\infty \frac{x^2 \mathrm{arcta\,n}(x)}{1+e^{\pi x}}dx$$

$$\triangle=A+B$$

$$*\ A=\int_0^\infty \frac{x^2 log(x)}{1+e^{\pi x}}dx=\int_0^\infty\left(\sum_{n=1}^\infty(-1)^{n-1}e^{-n\pi x}\right)x^2 log(x)dx$$

$$A=\sum_{n=1}^\infty(-1)^{n-1}\int_0^\infty x^2 log(x)e^{-n\pi x}dx, let: x=\frac{t}{n\pi}\to dx=\frac{dt}{n\pi}$$

$$\overset{x=t}{\Longrightarrow}A=\frac{1}{\pi^3}\sum_{n=1}^\infty\frac{(-1)^{n-1}}{n^3}\int_0^\infty x^2 log\left(\frac{x}{n\pi}\right)e^{-x}dx$$

$$A=\frac{1}{\pi^3}\left(\sum_{n=1}^\infty\frac{(-1)^{n-1}}{n^3}\int_0^\infty x^2 log(x)e^{-x}dx-\sum_{n=1}^\infty\frac{(-1)^{n-1}\mathrm{lo\,g}(n\pi)}{n^3}\int_0^\infty x^2e^{-x}dx\right)$$

$$\#Note: Gamma\ Function: \Gamma(z)=\int_0^\infty x^{z-1}e^{-x}dx\to\Gamma'(z)=\int_0^\infty x^{z-1}log(x)e^{-x}dx$$

$$A=\frac{1}{\pi^3}\left(\Gamma'(3)\sum_{n=1}^\infty\frac{(-1)^{n-1}}{n^3}-\Gamma(3)\sum_{n=1}^\infty\frac{(-1)^{n-1}[\log(\pi)+\log(n)]}{n^3}\right)$$

$$A=\frac{1}{\pi^3}\left(\Gamma'(3)\sum_{n=1}^\infty\frac{(-1)^{n-1}}{n^3}-2\mathrm{lo\,g}(\pi)\sum_{n=1}^\infty\frac{(-1)^{n-1}}{n^3}-2\sum_{n=1}^\infty\frac{(-1)^{n-1}\mathrm{lo\,g}(n)}{n^3}\right)$$

$\#Note$:

$$\Gamma'(z)=\Gamma(z)\psi^{(0)}(z)\to\Gamma'(3)=\Gamma(3)\psi^{(0)}(3)=2\left(\frac{3}{2}-\gamma\right)=3-2\gamma$$

$$Dirichlet\ Eta\ Function: \eta(s)=\sum_{n=1}^\infty\frac{(-1)^{n-1}}{n^s}\to\eta'(s)=-\sum_{n-1}^\infty\frac{(-1)^{n-1}log(n)}{n^s}$$

$$\eta(s)=(1-2^{1-s})\zeta(s)\to\eta'(s)=2^{1-s}log(2)\zeta(s)+(1-2^{1-s})\zeta'(s)$$

$$A=\frac{1}{\pi^3}\left(\eta(3)(3-2\gamma)-2\log(\pi)\,\eta(3)-2\eta'(3)\right)$$

$$A=\frac{1}{\pi^3}\left[\frac{3}{4}\zeta(3)(3-2\gamma)-\frac{3}{2}\zeta(3)log(\pi)+2\left(\frac{1}{4}log(2)\zeta(3)+\frac{3}{4}\zeta'(3)\right)\right]$$

$$A=\frac{\zeta(3)}{4\pi^3}\left(9-6\gamma+log\left(\frac{4}{\pi^6}\right)\right)+\frac{3\zeta'(3)}{2\pi^3}\therefore$$

$$*\,B=\int_0^\infty\frac{x^2\arctan(x)}{1+e^{\pi x}}dx=\int_0^\infty\left(\frac{1}{e^{\pi x}-1}-\frac{2}{e^{2\pi x}-1}\right)x^2arctang(x)dx$$

$$B=\int_0^\infty\frac{x^2arctan(x)}{e^{\pi x}-1}dx-2\int_0^\infty\frac{x^2arctan(x)}{e^{2\pi x}-1}dx$$

$$In:A=\int_0^\infty\frac{x^2\arctan(x)}{e^{\pi x}-1}dx,\ let:x=2t\to dx=2dt\overset{x=t}{\Longrightarrow}A$$

$$=8\int_0^\infty\frac{x^2\,arctan(2x)}{e^{2\pi x}-1}dx$$

$$B=8\int_0^\infty\frac{x^2arctan(2x)}{e^{2\pi x}-1}dx-2\int_0^\infty\frac{x^2arctan(x)}{e^{2\pi x}-1}dx=8A-2C$$

$$Let:I(k)=\int_0^\infty\frac{x^2\,arctan(kx)}{e^{2\pi x}-1}dx\to I'(k)=\int_0^\infty\frac{x^3}{(1+k^2x^2)(e^{2\pi x}-1)}dx$$

$$I'(k)=\frac{1}{k^2}\int_0^\infty\frac{x}{e^{2\pi x}-1}dx-\frac{1}{k^4}\int_0^\infty\frac{x}{\left(\frac{1}{k^2}+x^2\right)(e^{2\pi x}-1)}dx$$

#Note:

$$Binet\acute{}s\ Second\ Integral:\int_0^\infty\frac{x}{(z^2+x^2)(e^{2\pi x}-1)}dx$$

$$=\frac{1}{2}\left[log(z)-\frac{1}{2z}-\ \psi^{(0)}(z)\right]$$

$$\int_0^\infty\frac{x^{s-1}}{e^{2\pi x}-1}dx=\frac{\Gamma(s)\zeta(s)}{(2\pi)^s}$$

$$I'(k)=\frac{1}{24k^2}-\frac{1}{k^4}\left[\frac{1}{2}\left(\log\left(\frac{1}{k}\right)-\frac{k}{2}-\psi^{(0)}\left(\frac{1}{k}\right)\right)\right]$$

$$\int I'(k)dk=\frac{1}{24}\int\frac{dk}{k^2}-\frac{1}{2}\int\frac{1}{k^4}\log\left(\frac{1}{k}\right)dk+\frac{1}{4}\int\frac{dk}{k^3}+\frac{1}{2}\int\frac{1}{k^4}\psi^{(0)}\left(\frac{1}{k}\right)dk$$

$$In:\int\frac{1}{k^4}\psi^{(0)}\left(\frac{1}{k}\right)dk,\ let: t=\frac{1}{k}\to dt=-\frac{dx}{t^2}$$

$$-\int t^2\psi^{(0)}(t)dt\overset{I.B.P}{\Longrightarrow}-\left(t^2\log(\Gamma(t))-2\int t\log(\Gamma(t))dt\right)$$

$$\int\frac{1}{k^4}\psi^{(0)}\left(\frac{1}{k}\right)dk=-\frac{1}{k^2}\log\left(\Gamma\left(\frac{1}{k}\right)\right)+2\int_0^{\frac{1}{k}}t\log(\Gamma(t))dt$$

$$\Rightarrow I(k)=-\frac{1}{24k}-\frac{\log(k)}{6k^3}-\frac{1}{18k^3}-\frac{1}{8k^2}+\frac{1}{2}\left(-\frac{\log\left(\Gamma\left(\frac{1}{k}\right)\right)}{k^2}+2\int_0^{\frac{1}{k}}t\log(\Gamma(t))dt\right)+C$$

$$I(k)=-\frac{1}{24k}-\frac{\log(k)}{6k^3}-\frac{1}{18k^3}-\frac{1}{8k^2}-\frac{\log\left(\Gamma\left(\frac{1}{k}\right)\right)}{2k^2}+\int_0^{\frac{1}{k}}t\log(\Gamma(t))dt+C=P(k)+C$$

$$If: k\longrightarrow\infty\Rightarrow\arctan(kx)=\frac{\pi}{2}\wedge P(\infty)=0$$

$$\Rightarrow I(\infty)=\frac{\pi}{2}\int_0^{\infty}\frac{x^2}{e^{2\pi x}-1}dx=\frac{\pi}{2}\left(\frac{\zeta(3)}{4\pi^3}\right)=\frac{\zeta(3)}{8\pi^2}\to I(\infty)=P(\infty)+C\to\frac{\zeta(3)}{8\pi^2}=C$$

$\therefore$

$$I(k)=-\frac{1}{24k}-\frac{\log(k)}{6k^3}-\frac{1}{18k^3}-\frac{1}{8k^2}-\frac{\log\left(\Gamma\left(\frac{1}{k}\right)\right)}{2k^2}+\int_0^{\frac{1}{k}}t\log(\Gamma(t))dt+\frac{\zeta(3)}{8\pi^2}$$

$$\#I(1)=-\frac{2}{9}+\frac{\zeta(3)}{8\pi^2}+\int_0^1 t\log(\Gamma(t))dt$$

$$\#I(2)=-\frac{17}{288}-\frac{1}{48}log(2)-\frac{1}{16}log(\pi)+\frac{\zeta(3)}{8\pi^2}+\int_0^{\frac{1}{2}}tlog(\Gamma(t))dt$$

$$Let: Y(x)=\int_0^x tlog(\Gamma(t))dt\,, 0<x\leq 1$$

$$\#Note: log(\Gamma(x))=\left(\frac{1}{2}-x\right)(\gamma+log(2))+(1-x)log(\pi)-\frac{1}{2}log(sen(\pi x))$$

$$+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\log(n)}{n}sen(2\pi nx)$$

$$Y(x)=(\gamma+log(2))\int_0^x t\left(\frac{1}{2}-t\right)dt$$

$$+log(\pi)\int_0^x t(1-t)dt-\frac{1}{2}\int_0^x tlog(sen(\pi t))dt$$

$$+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\log(n)}{n}\int_0^x tsen(2\pi nt)dt$$

$$Y(x)=(\gamma+log(2))\left(\frac{x^2}{4}-\frac{x^3}{3}\right)+log(\pi)\left(\frac{x^2}{2}-\frac{x^3}{3}\right)-\frac{1}{2}\int_0^x tlog(sen(\pi t))dt$$

$$+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\log(n)}{n}\left(\frac{sen(2\pi nx)}{4\pi^2n^2}-\frac{xcos(2\pi nx)}{2\pi n}\right)$$

$$\#Note: log(sen(\pi x))=-log(2)-\sum_{k=1}^{\infty}\frac{cos(2k\pi x)}{k}$$

$$Y(x)=\frac{x^2}{4}\left(\gamma+log(2\pi^2)\right)-\frac{x^3}{3}(\gamma+log(2\pi))$$

$$-\frac{1}{2}\left(-log(2)\int_0^x tdt-\sum_{k=1}^{\infty}\frac{1}{k}\int_0^x tcos(2k\pi x)dt\right)+\frac{1}{4\pi^3}\sum_{n=1}^{\infty}\frac{log(n)sen(2\pi nx)}{n^3}$$

$$-\frac{x}{2\pi^2}\sum_{n=1}^{\infty}\frac{log(n)cos(2\pi nx)}{n^2}$$

$$Y(x)=\frac{x^2}{4}\left(\gamma+log(2\pi^2)\right)-\frac{x^3}{3}(\gamma+log(2\pi))$$

$$-\frac{1}{2}\left(-\frac{1}{2}log(2)x^2-\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{xsen(2\pi kx)}{2\pi k}+\frac{cos(2\pi kx)}{4\pi^2k^2}-\frac{1}{4\pi^2k^2}\right)\right)$$

$$+\frac{1}{4\pi^3}\sum_{n=1}^{\infty}\frac{log(n)sen(2\pi nx)}{n^3}-\frac{x}{2\pi^2}\sum_{n=1}^{\infty}\frac{log(n)cos(2\pi nx)}{n^2}$$

$$Y(x)=\frac{x^2}{4}\left(\gamma+log(4\pi^2)\right)-\frac{x^3}{3}(\gamma+log(2\pi))$$

$$+\frac{x}{4\pi}\sum_{k=1}^{\infty}\frac{sen(2\pi kx)}{k^2}+\frac{1}{8\pi^2}\sum_{k=1}^{\infty}\frac{cos(2\pi kx)}{k^3}-\frac{1}{8\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^3}$$

$$+\frac{1}{4\pi^3}\sum_{n=1}^{\infty}\frac{log(n)sen(2\pi nx)}{n^3}-\frac{x}{2\pi^2}\sum_{n=1}^{\infty}\frac{log(n)cos(2\pi nx)}{n^2}$$

$$\Rightarrow Y(1)=\frac{1}{4}\left(\gamma+log(4\pi^2)\right)-\frac{1}{3}(\gamma+log(2\pi))-\frac{1}{2\pi^2}\sum_{n=1}^{\infty}\frac{log(n)}{n^2}$$

$$Y(1)=-\frac{\gamma}{12}+\frac{1}{4}log(4\pi^2)-\frac{1}{3}\log(2\pi)+\frac{1}{2\pi^2}\zeta'(2)$$

$$Y(1)=-\frac{\gamma}{12}+\frac{1}{4}log(4\pi^2)-\frac{1}{3}\log(2\pi)+\frac{1}{2\pi^2}\left[\frac{\pi^2}{6}(\gamma-log(2\pi)-12log(A))\right]$$

$$Y(1)=\frac{1}{4}log\left(\frac{2\pi}{A^4}\right)$$

$$\#I(1)=-\frac{2}{9}+\frac{\zeta(3)}{8\pi^2}+\frac{1}{4}log\left(\frac{2\pi}{A^4}\right)\therefore$$

$$\Rightarrow Y\left(\frac{1}{2}\right)=\frac{\gamma}{48}+\frac{1}{12}log(2\pi)-\frac{1}{48}log(2)-\frac{7\zeta(3)}{32\pi^2}-\frac{\zeta'(2)}{8\pi^2}$$

$$Y\left(\frac{1}{2}\right)=\frac{\gamma}{48}+\frac{1}{12}log(2\pi)-\frac{1}{48}log(2)-\frac{7\zeta(3)}{32\pi^2}-\frac{1}{8\pi^2}\left[\frac{\pi^2}{6}(\gamma-log(2\pi)-12log(A))\right]$$

$$Y\left(\frac{1}{2}\right)=\frac{1}{96}log(16\pi^6A^{24})-\frac{7\zeta(3)}{32\pi^2}$$

$$\#I(2)=-\frac{17}{288}-\frac{1}{48}log(2)-\frac{1}{16}log(\pi)+\frac{\zeta(3)}{8\pi^2}+\frac{1}{96}log(16\pi^6A^{24})-\frac{7\zeta(3)}{32\pi^2}\therefore$$

$$\Rightarrow B=8\left(-\frac{17}{288}-\frac{1}{48}log(2)-\frac{1}{16}log(\pi)+\frac{\zeta(3)}{8\pi^2}+\frac{1}{96}log(16\pi^6A^{24})-\frac{7\zeta(3)}{32\pi^2}\right)$$

$$-2\left(-\frac{2}{9}+\frac{\zeta(3)}{8\pi^2}+\frac{1}{4}log\left(\frac{2\pi}{A^4}\right)\right)$$

$$B=4\,log(A)-\frac{\zeta(3)}{\pi^2}-\frac{1}{36}-\frac{1}{3}log(2)-\frac{1}{2}\log(\pi)\;\therefore$$

$Therefore$:

$$\triangle=\frac{\zeta(3)}{4\pi^3}\left(9-6\gamma+log\left(\frac{4}{\pi^6}\right)\right)+\frac{3\zeta'(3)}{2\pi^3}+4\,log(A)-\frac{\zeta(3)}{\pi^2}-\frac{1}{36}-\frac{1}{3}log(2)-\frac{1}{2}\log(\pi)$$

$$\triangle=\frac{\zeta(3)}{4\pi^3}\left(9-6\gamma+log\left(\frac{4}{\pi^6}\right)-4\pi\right)+\frac{3\zeta'(3)}{2\pi^3}+\frac{1}{6}log\left(\frac{A^{24}}{4\pi^3}\right)-\frac{1}{36}\;\therefore$$