ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\triangle = \int_0^1 \frac{x\ln^2(x)}{(1+3x)(2+x)}dx$$

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$$\triangle = -\frac{1}{5}\int_0^1 \frac{\ln^2(x)}{1+3x}dx + \frac{2}{5}\int_0^1 \frac{\ln^2(x)}{x+2}dx$$

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$$\#Note: \frac{1}{1+ax} = \sum_{n=0}^{\infty}(-a)^n x^n$$

$$\triangle = -\frac{1}{5}\sum_{n=0}^{\infty}(-3)^n \int_0^1 x^n \ln^2(x)\,dx + \frac{1}{5}\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n \int_0^1 x^n \ln^2(x)\,dx$$

$$\#Note: \int_0^1 x^p \ln^q(x)dx = \frac{(-1)^q q!}{(p+1)^{q+1}}$$

$$\triangle = -\frac{2}{5}\sum_{n=0}^{\infty}\frac{(-3)^n}{(n+1)^3} + \frac{2}{5}\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)^n}{(n+1)^3}$$

$$\xrightarrow{n+1=k\to n=k-1} \triangle = \frac{2}{15}\sum_{k=1}^{\infty}\frac{(-3)^k}{k^3} - \frac{4}{5}\sum_{k=1}^{\infty}\frac{\left(-\frac{1}{2}\right)^k}{k^3}$$

$$\#Note: Li_s(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^s}\ \textit{Polylogarithm Function}$$

$$\triangle = \frac{2}{15}Li_3(-3) - \frac{4}{5}Li_3\left(-\frac{1}{2}\right)$$

$$\#Note: Li_3(z) = Li_3\left(\frac{1}{z}\right) - \frac{1}{6}\ln^3(-z) - \frac{\pi^2}{6}\ln(-z)$$

$$\overset{z=-3}{\Longrightarrow} Li_3(-3) = Li_3\left(-\frac{1}{3}\right) - \frac{1}{6}\ln^3(3) - \frac{\pi^2}{6}\ln(3)$$

$$\triangle = \frac{2}{15}\left(\mathrm{Li}_3\left(-\frac{1}{3}\right) - \frac{1}{6}\ln^3(3) - \frac{\pi^2}{6}\ln(3)\right) - \frac{4}{5}\mathrm{Li}_3\left(-\frac{1}{2}\right)$$

$$\triangle = \frac{1}{45}\left(6\mathrm{Li}_3\left(-\frac{1}{3}\right) - \ln(3)\left(\ln^2(3) + \pi^2\right) - 36\mathrm{Li}_3\left(-\frac{1}{2}\right)\right) \therefore$$