ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\mathbf{I}=\int_0^1 \mathbf{x}(1-\mathbf{x}^2)\mathrm{Li}_3(-\mathbf{x}^2)\mathbf{dx}$$

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Solution 1 by Yang Silva Cartolin-Peru

$$\mathrm{I}=\int_0^1 \mathrm{x}\,\mathrm{Li}_3(-\mathrm{x}^2)\mathrm{dx}-\int_0^1 \mathrm{x}^3\mathrm{Li}_3(-\mathrm{x}^2)\mathrm{dx}$$

$$I=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\int_0^1 x^{2n+1}dx-\sum_{k=1}^{\infty}\frac{(-1)^k}{k^3}\int_0^1 x^{2k+3}dx$$

$$I=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\int_0^1 x^{2n+1}dx-\sum_{k=1}^{\infty}\frac{(-1)^k}{k^3}\int_0^1 x^{2k+3}dx$$

$$I=\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3(n+1)}-\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^3(k+2)}$$

$$I=\frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n}-\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}-\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)}\right)$$

$$-\frac{1}{2}\left(\frac{1}{8}\sum_{k=1}^{\infty}\frac{(-1)^k}{k}-\frac{1}{4}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}+\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^3}-\frac{1}{8}\sum_{k=1}^{\infty}\frac{(-1)^k}{(k+2)}\right)$$

$$I=\frac{7}{16}\sum_{k=1}^{\infty}\frac{(-1)^k}{k}+\frac{3}{8}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2}-\frac{1}{4}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3}-\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^k}{(k+1)}$$
$$+\frac{1}{16}\sum_{k=1}^{\infty}\frac{(-1)^k}{(k+2)}$$

$$I=-\frac{7}{16}\log(2)+\frac{3}{8}\eta(2)-\frac{1}{4}\eta(3)-\frac{1}{2}\sum_{j=2}^{\infty}\frac{(-1)^{k-1}}{k}-\frac{1}{16}\sum_{j=3}^{\infty}\frac{(-1)^{k-1}}{k}$$

$$I=-\frac{7}{16}\log(2)+\frac{3}{8}\left(\frac{1}{2}\zeta(2)\right)-\frac{1}{4}\left(\frac{3}{4}\zeta(3)\right)-\frac{1}{2}(\log(2)-1)-\frac{1}{16}\left(\log(2)-\frac{1}{2}\right)$$

$$I=\frac{\pi^2}{32}-\frac{3}{16}\zeta(3)-\log(2)+\frac{17}{32}$$

Solution 2 by Yang Silva Cartolin-Peru

$$I=\int_0^1 x\,Li_3(-x^2)dx-\int_0^1 x^3Li_3(-x^2)dx$$

$$Let: A=\int_0^1 x\,Li_3(-x^2)dx$$

$$IBP: u=Li_3(-x^2)\to du=\frac{2}{x}Li_2(-x^2)dx,\quad dv=xdx\to v=\frac{x^2}{2}$$

$$A=\left[\frac{x^2}{2}Li_3(-x^2)\right]_0^1-\int_0^1 x\,Li_2(-x^2)dx=\frac{1}{2}\left(-\frac{3}{4}\zeta(3)\right)-\int_0^1 x\,Li_2(-x^2)dx$$

$$IBP: u=Li_2(-x^2)\to du=-\frac{2}{x}ln(1+x^2)dx,\quad dv=xdx\to v=\frac{x^2}{2}$$

$$A=-\frac{3}{8}\zeta(3)-\left(\left[\frac{x^2}{2}Li_2(-x^2)\right]_0^1+\int_0^1 x\,ln(1+x^2)dx\right)$$

$$A=-\frac{3}{8}\zeta(3)-\frac{1}{2}\left(-\frac{\pi^2}{12}\right)-\int_0^1 x\left(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}x^{2k}\right)dx$$

$$A=-\frac{3}{8}\zeta(3)+\frac{\pi^2}{24}-log(2)+\frac{1}{2}\therefore$$

$$\text{Let: } B=\int_0^1 x^3Li_3(-x^2)dx$$

$$\text{IBP: } u=Li_3(-x^2)\to du=\frac{2}{x}Li_2(-x^2)dx,\quad dv=x^3dx\to v=\frac{x^4}{4}$$

$$B=-\frac{3}{16}\zeta(3)-\frac{1}{2}\int_0^1 x^3Li_2(-x^2)dx$$

$$\text{IBP: } u=Li_2(-x^2)\to du=-\frac{2}{x}ln(1+x^2)dx,\quad dv=x^3dx\to v=\frac{x^4}{4}$$

$$B=-\frac{3}{16}\zeta(3)-\frac{1}{2}\left(-\frac{\pi^2}{48}+\frac{1}{2}\int_0^1 x^3ln(1+x^2)dx\right)$$

$$B=-\frac{3}{16}\zeta(3)+\frac{\pi^2}{96}-\frac{1}{8}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k(k+2)}$$

$$B=-\frac{3}{16}\zeta(3)+\frac{\pi^2}{96}-\frac{1}{16}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}+\frac{1}{16}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k+2}$$

$$B=-\frac{3}{16}\zeta(3)+\frac{\pi^2}{96}-\frac{1}{16}log(2)+\frac{1}{16}\left(log(2)-\frac{1}{2}\right)=-\frac{3}{16}\zeta(3)+\frac{\pi^2}{96}-\frac{1}{32}\therefore$$

$$I=A-B\Rightarrow I=\frac{\pi^2}{32}-\frac{3}{16}\zeta(3)-log(2)+\frac{17}{32}\therefore$$