ROMANIAN MATHEMATICAL MAGAZINE

**Prove that:**

$$\int_0^1\int_0^1 \frac{\ln\left((1+xy)\sqrt{(1+x^2y^2)^3}\right)}{1+x^2y^2}\,dxdy = \frac{7\ln(2)\,G}{2} - \frac{\pi^3}{16}$$

$$G = \beta(2)\ is\ Catalan's\ constant$$

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$$\int_0^1\int_0^1 \frac{\ln\left((1+xy)\sqrt{(1+x^2y^2)^3}\right)}{1+x^2y^2}\,dx\,dy = I$$

$$\iint_{[0,1]^2} f(xy)\,dx\,dy = -\int_0^1 f(t)\ln t\,dt, I = -\int_0^1 \frac{\ln(1+t)+\frac{3}{2}\ln(1+t^2)}{1+t^2}\ln t\,dt$$

$$I_1 = \int_0^1 \frac{\ln(1+t)\ln t}{1+t^2}\,dt, \qquad A = \int_0^1 \frac{\ln^2 t}{1+t^2}\,dt \quad (1)$$

$$B = \int_0^1 \frac{\ln t\ln(1-t)}{1+t^2}\,dt \quad (2)$$

$$C = \int_0^1 \frac{\ln^2(1+t)}{1+t^2}\,dt \quad (3)$$

**Computing A** In (1) let $t \to \frac{1-t}{1+t}$:

$$A = \int_0^1 \frac{\ln^2\left(\frac{1-t}{1+t}\right)}{1+t^2}\,dt = \int_0^1 \frac{\ln^2(1-t) - 2\ln(1-t)\ln(1+t) + \ln^2(1+t)}{1+t^2}\,dt$$

$$A = \int_0^1 \frac{\ln^2(1-t)}{1+t^2}\,dt - 2\int_0^1 \frac{\ln(1-t)\ln(1+t)}{1+t^2}\,dt + \int_0^1 \frac{\ln^2(1+t)}{1+t^2}\,dt$$

$$A = \int_0^1 \frac{\ln^2 t}{1+t^2}\,dt, \qquad \int_0^1 \frac{\ln^{n-1} t}{1+t^2}\,dt = \frac{\beta(n)\Gamma(n)}{(-1)^{n-1}}$$

$$A = \frac{\beta(3)\Gamma(3)}{(-1)^2} = \frac{\pi^3}{16}$$

$$\int_0^1 \frac{\ln^2(1-t)}{1+t^2}\,dt, \qquad \int_0^1 \frac{\ln^n(1-t)}{1+t^2}\,dt = (-1)^n\,n!\,\Im\left(\mathrm{Li}_{n+1}\left(\frac{1+i}{2}\right)\right)$$

$$\int_0^1 \frac{\ln^2(1-t)}{1+t^2}\,dt = 2\,\Im\left(\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right)$$

Let

$$D=\int_0^1 \frac{\ln(1-t)\ln(1+t)}{1+t^2}\,dt, \qquad E=\int_0^1 \frac{\ln(t)\ln(1+t)}{1+t^2}\,dt$$

Then

$$C-2D=\frac{\pi^3}{16}-2\,\Im\left(\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right) \quad (4)$$

Relation (5), from B

In (2) $B=\int_0^1 \frac{\ln t\ln(1-t)}{1+t^2}\,dt$, let $t\to\frac{1-t}{1+t}$:

$$B=\int_0^1 \frac{\ln\left(\frac{1-t}{1+t}\right)\ln\left(\frac{2t}{1+t}\right)}{1+t^2}\,dt$$

$$B=\ln 2\int_0^1 \frac{\ln\left(\frac{1-t}{1+t}\right)}{1+t^2}\,dt-D+B+C-E$$

$$C-D-E=G\ln 2 \quad (5)$$

Relation (6), from C In (3)

$$C=\left(\int_0^\infty-\int_1^\infty\right)\frac{\ln^2(1+t)}{1+t^2}\,dt, C=\int_0^\infty \frac{\ln^2(1+t)}{1+t^2}\,dt-\int_1^\infty \frac{\ln^2(1+t)}{1+t^2}\,dt$$

let $t\to\frac{1}{t}$:

$$C=\int_0^\infty \frac{\ln^2(1+t)}{1+t^2}\,dt-\left[\int_0^1 \frac{\ln^2(1+t)-2\ln t\ln(1+t)+\ln^2 t}{1+t^2}\,dt\right]$$

$$C=\int_0^\infty \frac{\ln^2(1+t)}{1+t^2}\,dt-C+2E-\frac{\pi^3}{16}, C-E=\frac{1}{2}\int_0^\infty \frac{\ln^2(1+t)}{1+t^2}\,dt-\frac{\pi^3}{32} \quad (6)$$

$$\int_0^\infty \frac{\ln^n(1+t)}{1+t^2}\,dt=(-1)^n\,n!\,\Im\left(\mathrm{Li}_{n+1}(1+i)\right)$$

$$\int_0^\infty \frac{\ln^2(1+t)}{1+t^2}\,dt=2\,\Im\left(\mathrm{Li}_3(1+i)\right), \qquad C-E=\Im\left(\mathrm{Li}_3(1+i)\right)-\frac{\pi^3}{32} \quad (6)$$

Solving for D, C and , $C-2D=\frac{\pi^3}{16}-2\,\Im\left(\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right)$ (4)

$$C-D-E=G\ln(2) \quad (5), \qquad -E=\Im\left(\mathrm{Li}_3(1+i)\right)-\frac{\pi^3}{32} \quad (6)$$

(5) and $-1\cdot$(6) sum:

$$D=\Im\left(\mathrm{Li}_3(1+i)\right)-\frac{\pi^3}{32}-G\ln(2),\quad \text{in (4), } C=2D+\frac{\pi^3}{16}-2\,\Im\left(\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right)$$

$$C=-4\,\Im\left(\mathrm{Li}_3\left(\frac{1+i}{2}\right)\right)-2\ln(2)\,G+\frac{7\pi^3}{64}+\frac{3\pi}{16}\ln^2 2$$

$$E=C-\Im\left(\mathrm{Li}_3(1+i)\right)-\frac{\pi^3}{32}$$

$$I_1=E=\int_0^1\frac{\ln(t)\ln(1+t)}{1+t^2}\,dt=3\,\Im\left(\mathrm{Li}_3(1+i)\right)-\frac{5\pi^3}{64}-\frac{3\pi\ln^2(2)}{16}-2\ln(2)\,G$$

**Computing $I_2$**

$$I_2=\int_0^1\frac{\ln(t)\ln(1+t^2)}{1+t^2}\,dt,\qquad \frac{\ln(1+t^2)}{1+t^2}=\sum_{n=1}^{\infty}(-1)^{n+1}H_n\,t^{2n}$$

$$I_2=\sum_{n=1}^{\infty}(-1)^{n+1}H_n\int_0^1 t^{2n}\ln(t)\,dt=\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{(2n+1)^2}$$

$$\int_0^1\frac{t^{2n}}{1+t}\,dt=\ln(2)+H_n-H_{2n},\qquad H_{2n+1}=\frac{1}{2n+1}+H_{2n}$$

$$H_n=-\ln(2)+H_{2n+1}-\frac{1}{2n+1}+\int_0^1\frac{t^{2n}}{1+t}\,dt$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}H_n=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\left[-\ln(2)+H_{2n+1}-\frac{1}{2n+1}+\int_0^1\frac{t^{2n}}{1+t}\,dt\right]$$

$$I_2=-\ln(2)\,G+\sum_{n=0}^{\infty}\frac{(-1)^n H_{2n+1}}{(2n+1)^2}-\beta(3)+\int_0^1\frac{1}{t(1+t)}\left(\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n+1}}{(2n+1)^2}\right)dt$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n H_{2n+1}}{(2n+1)^2}=\Im\left(\sum_{n=1}^{\infty}\frac{i^n H_n}{n^2}\right)$$

$$\sum_{n=1}^{\infty}\frac{t^n H_n}{n^2}=\mathrm{Li}_3(t)-\mathrm{Li}_3(1-t)+\mathrm{Li}_2(1-t)\ln(1-t)+\frac{1}{2}\ln(t)\ln^2(1-t)$$
$$+\zeta(3),\qquad t=i$$

$$\Im\left(\mathrm{Li}_3(i)\right)=\frac{\pi^3}{32},\qquad \Im\left(\ln(1-i)\,\mathrm{Li}_2(1-i)\right)=-\frac{\ln(2)\,G}{2}-\frac{\pi^3}{64}-\frac{\pi\ln^2(2)}{8}$$

$$\Im\left(\frac{1}{2}\ln(i)\ln^2(1-i)\right)=-\frac{\pi^3}{64}+\frac{\pi}{16}\ln^2 2,$$

$$S_1=-\Im\big(\mathrm{Li}_3(1-i)\big)-\frac{\pi\ln^2 2}{8}-\frac{\pi\ln^2 2}{16}$$

$$\circledast\quad \int_0^1\frac{1}{1+t}\,\Im\left(\frac{\mathrm{Li}_2(t)}{t}\right)dt=\int_0^1\frac{1}{1+t}\left(\int_0^1\frac{-i\ln x}{1-ixt}\,dx\right)dt,\qquad \text{let } xt=u$$

$$\int_0^1\int_0^t\frac{\ln\left(\frac{t}{u}\right)}{t(t+1)(1+u^2)}\,du\,dt=K$$

$$K=\int_0^1\frac{1}{1+u^2}\left(\mathrm{Li}_2(-u)+\frac{1}{2}\ln^2 u+\ln(2)\ln u+\frac{\pi^2}{12}\right)du$$

$$K=\int_0^1\frac{\mathrm{Li}_2(-u)}{1+u^2}\,du+\beta(3)+\ln(2)\,G+\frac{\pi^3}{48}$$

$$M=\int_0^1\frac{\mathrm{Li}_2(-u)}{1+u^2}\,du=\int_0^1\frac{1}{1+u^2}\left(\int_0^1\frac{u\ln x}{1+ux}\,dx\right)du$$

$$M=\int_0^1\ln x\left(\int_0^1\frac{u}{(1+ux)(1+u^2)}\,du\right)dx$$

$$M=\int_0^1\ln x\left(\frac{\pi x}{4(1+x^2)}+\frac{\ln(2)}{2(1+x^2)}-\frac{\ln(1+x)}{1+x^2}\right)dx$$

$$M=\frac{\pi}{16}\int_0^1\frac{\ln x}{1+x}\,dx+\frac{\ln(2)}{2}(-G)-E$$

$$M=\frac{7\pi^3}{96}+\frac{3\pi}{16}\ln^2 2+\frac{3}{2}\ln(2)\,G-3\,\Im\big(\mathrm{Li}_3(1+i)\big)$$

$$I_2=2\,\Im\big(\mathrm{Li}_3(1-i)\big)+\frac{3\pi^3}{32}+\frac{\pi}{8}\ln^2 2-\ln(2)\,G$$

**Conclusion** $I=-I_1-\frac{3}{2}I_2$

$$I=-3\,\Im\big(\mathrm{Li}_3(1+i)\big)+\frac{5\pi^3}{64}+\frac{3\pi}{16}\ln^2 2+2\ln(2)\,G$$
$$-\frac{3}{2}\left(2\,\Im\big(\mathrm{Li}_3(1-i)\big)+\frac{3\pi^3}{32}+\frac{\pi\ln^2 2}{8}-\ln(2)\,G\right)$$

$$I=-3\,\Im\big(\mathrm{Li}_3(1+i)+\mathrm{Li}_3(1-i)\big)+\left(\frac{5\pi^3}{64}-\frac{9\pi^3}{64}\right)+G\ln(2)\left(2+\frac{3}{2}\right)$$

$$\boxed{I=\frac{7}{2}\,G\ln(2)-\frac{\pi^3}{16}}$$

ROMANIAN MATHEMATICAL MAGAZINE

*Solution 2 by Pham Duc Nam- Vietnam*

$$I=\int_0^1\int_0^1\frac{\ln\left((1+xy)\sqrt{(1+x^2y^2)^3}\right)}{1+x^2y^2}\,dxdy=\frac{7}{2}G\ln(2)-\frac{\pi^3}{16}$$

1.) Reduce the given integral

By symmetry integral : $\int_0^1\int_0^1 f(xy)dxdy=-\int_0^1\ln(x)f(x)dx$

$$\Rightarrow I=-\int_0^1\ln(x)\left(\frac{\ln(1+x)}{1+x^2}+\frac{3}{2}\frac{\ln(1+x^2)}{1+x^2}\right)dx$$

$$=-\int_0^1\frac{\ln(x)\ln(1+x)}{1+x^2}dx-\frac{3}{2}\int_0^1\frac{\ln(x)\ln(1+x^2)}{1+x^2}dx=-A-\frac{3}{2}B$$

2.) Evaluating component integrals

2.1) Evaluating of $B$

$$B=\int_0^1\frac{\ln(x)\ln(1+x^2)}{1+x^2}dx\overset{x=\tan(t)}{=}-2\int_0^{\frac{\pi}{4}}\ln\left(\tan(t)\right)\ln\left(\cos(t)\right)dt$$

$$=-2\int_0^{\frac{\pi}{4}}\ln\left(\sin(t)\right)\ln\left(\cos(t)\right)dt+2\int_0^{\frac{\pi}{4}}\ln^2\left(\cos(t)\right)dt$$

$$=-\int_0^{\frac{\pi}{2}}\ln\left(\sin(t)\right)\ln\left(\cos(t)\right)dt+2\int_0^{\frac{\pi}{4}}\ln^2\left(\cos(t)\right)dt$$

Derivative of Beta function

$$=-\left(\frac{\pi}{2}\ln^2(2)-\frac{\pi^3}{48}\right)+2B_1$$

$$*\ B_1=\int_0^{\frac{\pi}{4}}\ln^2\left(\cos(t)\right)dt\ \because\ 0\le x<\frac{\pi}{2}:\ \mathrm{Log}(1+e^{2ix})=\ln\left(2\cos(x)\right)+ix$$

$$\Rightarrow\int_0^{\frac{\pi}{4}}\mathrm{Log}^2\left(1+e^{2ix}\right)dx=\int_0^{\frac{\pi}{4}}\left(\ln\left(2\cos(x)\right)+ix\right)^2dx$$

Taking real part we have :

$$B_1=\int_0^{\frac{\pi}{4}}\left(x^2-\ln^2(2)\right)dx-2\ln(2)\int_0^{\frac{\pi}{4}}\ln\left(\cos(x)\right)dx$$

$$+\Re\left\{\int_0^{\frac{\pi}{4}}\mathrm{Log}^2\left(1+e^{2ix}\right)dx\right\}$$

$$=\frac{\pi^3}{192}+\frac{\pi}{4}\ln^2(2)-G\ln(2)+\Re\left\{\int_0^{\frac{\pi}{4}}\text{Log}^2\left(1+e^{2ix}\right)dx\right\}$$

$$\text{And}:\Re\left\{\int_0^{\frac{\pi}{4}}\text{Log}^2\left(1+e^{2ix}\right)dx\right\}\overset{z=e^{2ix}}{=}\frac{1}{2}\Im\left\{\int_1^i\frac{\text{Log}^2(1+z)}{z}dz\right\}$$

By integration by parts :

$$\int\frac{\text{Log}^2(1+z)}{z}dz=$$

$$=-2\text{Li}_3(1+z)+2\text{Li}_2(1+z)\text{Log}(1+z)+\text{Log}(-z)\text{Log}^2(1+z)+C$$

$$\Rightarrow\frac{1}{2}\Im\left\{\int_1^i\frac{\ln^2(1+z)}{z}dz\right\}$$

$$=\frac{1}{2}\Im\{\ln(-i)\ln^2(1+i)+2\ln(1+i)\text{Li}_2(1+i)-2\text{Li}_3(1+i)\}$$

$$=\frac{\pi^3}{32}+\frac{\pi}{16}\ln^2(2)+\frac{1}{2}G\ln(2)-\Im\{\text{Li}_3(1+i)\}\Rightarrow B=-\left(\frac{\pi}{2}\ln^2(2)-\frac{\pi^3}{48}\right)+2B_1$$

$$=-2\Im\{\text{Li}_3(1+i)\}-G\ln(2)+\frac{3\pi^3}{32}+\frac{\pi}{8}\ln^2(2)$$

2.2) Evaluating of $A$

$$A=\int_0^1\frac{\ln(x)\ln(1+x)}{1+x^2}dx\overset{\text{IBPs}}{=}-\int_0^1\frac{\arctan(x)\ln(x)}{1+x}dx-\int_0^1\frac{\arctan(x)\ln(1+x)}{x}dx$$

$$\breve{N}\qquad\qquad\breve{L}$$

$$*\ \text{Let}:M=\int_0^1\frac{\arctan(t)}{t}\ln(1-t)dt=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\int_0^1t^{2n}\ln(1-t)dt$$

$$=-\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\frac{H_{2n+1}}{2n+1}=-\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}H_{2n+1}=-\Im\left\{\sum_{n=1}^{\infty}\frac{i^nH_n}{n^2}\right\}$$

By generating function :

$$\sum_{n=1}^{\infty}\frac{x^nH_n}{n^2}=$$

$$\text{Li}_3(x)-\text{Li}_3(1-x)+\ln(1-x)\text{Li}_2(1-x)+\frac{1}{2}\ln(x)\ln^2(1-x)+\zeta(3)$$

$$\Rightarrow M=\frac{\pi}{16}\ln^2(2)+\frac{1}{2}G\ln(2)+\Im\{\text{Li}_3(1-i)\}$$

ROMANIAN MATHEMATICAL MAGAZINE

Define a function : $g(x)=\int_0^x \frac{\operatorname{arctanh}(z)}{z}\mathrm{d}z \Rightarrow g(z)+g\left(\frac{1-z}{1+z}\right)=\frac{\pi^2}{8}-\frac{1}{2}\ln(z)\ln\left(\frac{1-z}{1+z}\right)$

$$\Rightarrow 2\int_0^1 \frac{g(x)}{1+x^2}\mathrm{d}x=\frac{\pi^3}{32}+\frac{1}{2}\int_0^1 \frac{1}{1+x^2}\ln(x)\ln\left(\frac{1+x}{1-x}\right)\mathrm{d}x$$

$$\overset{\text{IBPs}}{\Leftrightarrow} \frac{\pi^3}{16}-L+M=\frac{\pi^3}{32}+\frac{1}{2}\int_0^1 \frac{1}{1+x^2}\ln(x)\ln\left(\frac{1+x}{1-x}\right)\mathrm{d}x$$

$$\text{But : } \overset{P}{\breve{}}=\frac{1}{2}\int_0^1 \frac{1}{1+x^2}\ln(x)\ln\left(\frac{1+x}{1-x}\right)\mathrm{d}x=\int_0^1\int_0^1 \frac{z\ln(z)}{(1+z^2)(1-z^2t^2)}\mathrm{d}z\mathrm{d}t$$

$$=\int_0^1 \frac{1}{1+t^2}\int_0^1\left(\frac{t^2z\ln(z)}{1-z^2t^2}+\frac{z\ln(z)}{1+z^2}\right)\mathrm{d}z\mathrm{d}t$$

$$=\int_0^1 \frac{1}{1+t^2}\int_0^1 \frac{t^2z\ln(z)}{1-z^2t^2}\mathrm{d}z\mathrm{d}t+\int_0^1 \frac{1}{1+t^2}\mathrm{d}t\int_0^1 \frac{z\ln(z)}{1+z^2}\mathrm{d}z$$

$$=-\frac{\pi^3}{192}-\frac{1}{4}\int_0^1 \frac{\operatorname{Li}_2(t^2)}{1+t^2}\mathrm{d}t$$

$$\overset{\text{IBPs}}{=} -\frac{\pi^3}{192}-\frac{1}{4}\left(\frac{\pi}{4}\zeta(2)+2\int_0^1 \frac{\ln(1-t^2)\arctan(t)}{t}\mathrm{d}t\right)$$

$$\overset{L+M}{\breve{}}$$

$$\Rightarrow P=-\frac{\pi^3}{64}-\frac{L}{2}-\frac{M}{2}\Rightarrow \frac{\pi^3}{16}-L+M=\frac{\pi^3}{32}+\left(-\frac{\pi^3}{64}-\frac{L}{2}-\frac{M}{2}\right)\Leftrightarrow M=\frac{L}{3}-\frac{\pi^3}{32}$$

$$\Rightarrow L=3\left(M+\frac{\pi^3}{32}\right)=3\left(\frac{\pi}{16}\ln^2(2)+\frac{1}{2}G\ln(2)+\Im\{\operatorname{Li}_3(1-i)\}+\frac{\pi^3}{32}\right)$$

$$=\frac{3\pi}{16}\ln^2(2)+\frac{3}{2}G\ln(2)+3\Im\{\operatorname{Li}_3(1-i)\}+\frac{3\pi^3}{32}$$

$$*\ N=\int_0^1 \frac{\arctan(x)\ln(x)}{1+x}\mathrm{d}x=\int_0^1\int_0^x \frac{\ln(x)}{(1+x)(1+y^2)}\mathrm{d}y\mathrm{d}x$$

$$=\int_0^1\int_0^1 \frac{\ln(x)}{(1+x)(1+y^2)}\mathrm{d}x\mathrm{d}y-\int_0^1\int_0^y \frac{\ln(x)}{(1+x)(1+y^2)}\mathrm{d}x\mathrm{d}y$$

$$=-\frac{\pi^3}{48}-\left(-\frac{\pi^3}{192}-\frac{1}{2}G\ln(2)\right)=\frac{1}{2}G\ln(2)-\frac{\pi^3}{64}$$

$$\text{Where}: \int_0^1\int_0^y \frac{\ln(x)}{(1+x)(1+y^2)}dxdy \overset{x\to xy}{=} \int_0^1\int_0^1 \frac{y\ln(xy)}{(1+xy)(1+y^2)}dxdy$$

$$\overset{\text{Symmetry}}{=} \frac{1}{2}\int_0^1\int_0^1 \frac{\ln(xy)}{1+xy}\left(\frac{y}{1+y^2}+\frac{x}{1+x^2}\right)dxdy$$

$$=\frac{1}{2}\int_0^1\int_0^1 \frac{(x+y)\ln(xy)}{(1+x^2)(1+y^2)}dxdy = -\frac{\pi^3}{192}-\frac{1}{2}G\ln(2) \Rightarrow A = -N - L$$

$$= -\frac{1}{2}G\ln(2)+\frac{\pi^3}{64}-\frac{3\pi}{16}\ln^2(2)-\frac{3}{2}G\ln(2)-3\Im\{\text{Li}_3(1-i)\}-\frac{3\pi^3}{32}$$

$$= -3\Im\{\text{Li}_3(1-i)\}-2G\ln(2)-\frac{5\pi^3}{64}-\frac{3\pi}{16}\ln^2(2)$$

3.) Conclusion

From above sections : $A = -3\Im\{\text{Li}_3(1-i)\}-2G\ln(2)-\frac{5\pi^3}{64}-\frac{3\pi}{16}\ln^2(2)$

$$B = -2\Im\{\text{Li}_3(1+i)\}-G\ln(2)+\frac{3\pi^3}{32}+\frac{\pi}{8}\ln^2(2) \Rightarrow I = -A-\frac{3}{2}B$$

$$= 3\Im\{\text{Li}_3(1-i)\}+2G\ln(2)+\frac{5\pi^3}{64}+\frac{3\pi}{16}\ln^2(2)$$
$$-\frac{3}{2}\left(-2\Im\{\text{Li}_3(1+i)\}-G\ln(2)+\frac{3\pi^3}{32}+\frac{\pi}{8}\ln^2(2)\right)$$

$$= -\frac{\pi^3}{16}+\frac{7}{2}G\ln(2)+3\Im\{\text{Li}_3(1+i)+\text{Li}_3(1-i)\} = \frac{7}{2}G\ln(2)-\frac{\pi^3}{16}, \text{ proved.}$$

$\because\ \Im\{\text{Li}_3(1+i)+\text{Li}_3(1-i)\} = 0$ since $1+i$ and $1-i$ is a pair of conjugate complex numbers.