

ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$I = \int_0^{\infty} \frac{x^2 \ln(\alpha x)}{e^{4x^2} - 1} dx, \forall \alpha > 0$$

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Solution by Nawar Alasadi-Iraq

$$\begin{aligned}
 I &= \int_0^{\infty} \frac{x^2 \ln(\alpha x)}{e^{4x^2} - 1} dx \\
 u &= 4x^2 \Rightarrow dx = \frac{du}{4\sqrt{u}}, \quad x = \frac{\sqrt{u}}{2} \\
 I &= \int_0^{\infty} \frac{\frac{u}{4} \ln\left(\frac{\alpha\sqrt{u}}{2}\right)}{e^u - 1} \frac{du}{4\sqrt{u}} \\
 &= \frac{1}{16} \int_0^{\infty} \frac{u^{1/2} \left(\ln\left(\frac{\alpha}{2}\right) + \frac{1}{2} \ln u \right)}{e^u - 1} du \\
 &= \frac{\ln\left(\frac{\alpha}{2}\right)}{16} \int_0^{\infty} \frac{u^{1/2}}{e^u - 1} du + \frac{1}{32} \int_0^{\infty} \frac{u^{1/2} \ln u}{e^u - 1} du \\
 \int_0^{\infty} \frac{t^{s-1}}{e^t - 1} dt &= \Gamma(s) \zeta(s) \\
 \int_0^{\infty} \frac{u^{1/2}}{e^u - 1} du &= \Gamma\left(\frac{3}{2}\right) \zeta\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2} \zeta\left(\frac{3}{2}\right) \\
 \int_0^{\infty} \frac{u^{1/2} \ln u}{e^u - 1} du &= \frac{\partial}{\partial s} [\Gamma(s) \zeta(s)]_{s=\frac{3}{2}} = \Gamma\left(\frac{3}{2}\right) \zeta\left(\frac{3}{2}\right) \psi\left(\frac{3}{2}\right) + \Gamma\left(\frac{3}{2}\right) \zeta'\left(\frac{3}{2}\right) \\
 \psi\left(\frac{3}{2}\right) &= 2 - \gamma - 2\ln 2 \\
 \int_0^{\infty} \frac{u^{1/2} \ln u}{e^u - 1} du &= \frac{\sqrt{\pi}}{2} \left[\zeta\left(\frac{3}{2}\right) (2 - \gamma - 2\ln 2) + \zeta'\left(\frac{3}{2}\right) \right] \\
 I &= \frac{\ln\left(\frac{\alpha}{2}\right)}{16} \left(\frac{\sqrt{\pi}}{2} \zeta\left(\frac{3}{2}\right) \right) + \frac{1}{32} \left(\frac{\sqrt{\pi}}{2} \left[\zeta\left(\frac{3}{2}\right) (2 - \gamma - 2\ln 2) + \zeta'\left(\frac{3}{2}\right) \right] \right) \\
 &= \frac{\sqrt{\pi}}{64} \left[2\ln\left(\frac{\alpha}{2}\right) \zeta\left(\frac{3}{2}\right) + \zeta\left(\frac{3}{2}\right) (2 - \gamma - 2\ln 2) + \zeta'\left(\frac{3}{2}\right) \right] \\
 &= \frac{\sqrt{\pi}}{64} \left[\zeta\left(\frac{3}{2}\right) (2\ln \alpha - 2\ln 2 + 2 - \gamma - 2\ln 2) + \zeta'\left(\frac{3}{2}\right) \right] \\
 &= \frac{\sqrt{\pi}}{64} \left[\zeta\left(\frac{3}{2}\right) (2\ln \alpha - 4\ln 2 - \gamma + 2) + \zeta'\left(\frac{3}{2}\right) \right]
 \end{aligned}$$