

ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$I = \int_0^{\infty} \frac{x \ln^2(\alpha x)}{e^{\sigma x} - 1} dx \quad \forall \alpha, \sigma > 0$$

Proposed by Samuel Ajibade-Nigeria

Solution by proposer

$$I = \int_0^{\infty} \frac{x \ln^2(\alpha x)}{e^{\sigma x} - 1} dx = \frac{\partial^2}{\partial \beta^2} \int_0^{\infty} \frac{x(\alpha x)^{\beta}}{e^{\sigma x} - 1} dx \Big|_{\beta=0} = \frac{\partial^2}{\partial \beta^2} \alpha^{\beta} \int_0^{\infty} \frac{x^{\beta+1}}{e^{\sigma x} - 1} dx \Big|_{\beta=0}$$

$$I = \frac{\partial^2}{\partial \beta^2} \alpha^{\beta} \int_0^{\infty} \sum_{k=1}^{\infty} e^{-\sigma k x} x^{\beta+1} dx \Big|_{\beta=0} \quad (\text{Geometric series: } |e^{-\sigma x}| < 1)$$

$$I = \frac{\partial^2}{\partial \beta^2} \sum_{k=1}^{\infty} \alpha^{\beta} \int_0^{\infty} x^{\beta+1} e^{-\sigma k x} dx \Big|_{\beta=0} \quad (\text{Fubini - Tonelli Theorem})$$

$$I = \frac{\partial^2}{\partial \beta^2} \sum_{k=1}^{\infty} \alpha^{\beta} \frac{\Gamma(\beta+2)}{(\sigma k)^{\beta+2}} \Big|_{\beta=0} \quad (\text{Laplace Transforms})$$

$$I = \frac{\partial^2}{\partial \beta^2} \frac{\alpha^{\beta} \Gamma(\beta+2)}{\sigma^{\beta+2}} \sum_{k=1}^{\infty} \frac{1}{k^{\beta+2}} \Big|_{\beta=0} = \frac{\partial^2}{\partial \beta^2} \frac{\alpha^{\beta} \Gamma(\beta+2) \zeta(\beta+2)}{\sigma^{\beta+2}} \Big|_{\beta=0}$$

$$= \frac{1}{\sigma^2} \frac{\partial^2}{\partial \beta^2} \left(\frac{\alpha}{\sigma} \right)^{\beta} \Gamma(\beta+2) \zeta(\beta+2) \Big|_{\beta=0}$$

$$I = \frac{1}{\sigma^2} \frac{\partial^2}{\partial \beta^2} \lambda^{\beta} \Gamma(\beta+2) \zeta(\beta+2) \Big|_{\beta=0} \quad \left(\frac{\alpha}{\sigma} = \lambda \right)$$

$$I = \frac{1}{\sigma^2} \frac{\partial^2}{\partial \beta^2} \lambda^{\beta} \Gamma(\beta+2) \zeta(\beta+2) \Big|_{\beta=0}$$

$$= \frac{1}{\sigma^2} \frac{\partial}{\partial \beta} \lambda^{\beta} \Gamma(\beta+2) \zeta(\beta+2) \left(\ln(\lambda) + \psi(\beta+2) + \frac{\zeta'(\beta+2)}{\zeta(\beta+2)} \right) \Big|_{\beta=0}$$

$$I = \frac{1}{\sigma^2} \frac{\partial}{\partial \beta} \lambda^{\beta} \Gamma(\beta+2) \zeta(\beta+2) \left(\ln(\lambda) + \psi(\beta+2) + \frac{\zeta'(\beta+2)}{\zeta(\beta+2)} \right) \Big|_{\beta=0} \quad \dots (0)$$

$$\frac{\partial}{\partial \beta} \lambda^{\beta} \Gamma(\beta+2) \zeta(\beta+2) \ln(\lambda)$$

$$= \lambda^{\beta} \Gamma(\beta+2) \zeta(\beta+2) \ln(\lambda) \left(\ln(\lambda) + \psi(\beta+2) + \frac{\zeta'(\beta+2)}{\zeta(\beta+2)} \right) \quad \dots (1)$$

ROMANIAN MATHEMATICAL MAGAZINE

$$\frac{\partial}{\partial \beta} \lambda^\beta \Gamma(\beta + 2) \zeta(\beta + 2) \psi(\beta + 2)$$

$$= \lambda^\beta \Gamma(\beta + 2) \zeta(\beta + 2) \psi(\beta + 2) \left(\ln(\lambda) + \frac{\psi'(\beta + 2)}{\psi(\beta + 2)} + \psi(\beta + 2) + \frac{\zeta'(\beta + 2)}{\zeta(\beta + 2)} \right) \dots (2)$$

$$\frac{\partial}{\partial \beta} \lambda^\beta \Gamma(\beta + 2) \zeta'(\beta + 2) = \lambda^\beta \Gamma(\beta + 2) \zeta'(\beta + 2) \left(\ln(\lambda) + \psi(\beta + 2) + \frac{\zeta''(\beta + 2)}{\zeta'(\beta + 2)} \right) \dots (3)$$

Evaluate equation 1 to 3 at $\beta = 0$, and substitute in equation 0

$$NB: \Gamma(2) = 1, \quad \zeta(2) = \frac{\pi^2}{6}, \quad \psi(2) = 1 - \gamma, \quad \psi'(2) = \frac{\pi^2}{6} - 1, \quad \zeta'(2) = \frac{\pi^2}{6} \ln\left(\frac{2\pi e^\gamma}{A^{12}}\right)$$

$$\ln(\lambda) \frac{\pi^2}{6} \left(\ln(\lambda) + 1 - \gamma + \ln\left(\frac{2\pi e^\gamma}{A^{12}}\right) \right) \dots (1)$$

$$\frac{\pi^2}{6} (1 - \gamma) \left(\ln(\lambda) + \frac{\pi^2 - 6}{6(1 - \gamma)} + 1 - \gamma + \ln\left(\frac{2\pi e^\gamma}{A^{12}}\right) \right) \dots (2)$$

$$\frac{\pi^2}{6} \ln\left(\frac{2\pi e^\gamma}{A^{12}}\right) \left(\ln(\lambda) + 1 - \gamma + \frac{6\zeta''(2)}{\pi^2 \ln\left(\frac{2\pi e^\gamma}{A^{12}}\right)} \right) \dots (3)$$

$$I = \frac{\pi^2}{6\sigma^2} \left(\ln^2(\lambda) + \ln(\lambda)(1 - \gamma) + \ln(\lambda) \ln\left(\frac{2\pi e^\gamma}{A^{12}}\right) + (1 - \gamma) \ln(\lambda) + \frac{\pi^2 - 6}{6} + (1 - \gamma)^2 + (1 - \gamma) \ln\left(\frac{2\pi e^\gamma}{A^{12}}\right) + \ln\left(\frac{2\pi e^\gamma}{A^{12}}\right) \ln(\lambda) + \ln\left(\frac{2\pi e^\gamma}{A^{12}}\right) (1 - \gamma) + \frac{6\zeta''(2)}{\pi^2} \right)$$

$$I = \frac{\pi^2}{6\sigma^2} \left(\ln^2(\lambda) + 2\ln(\lambda) \left(1 - \gamma + \ln\left(\frac{2\pi e^\gamma}{A^{12}}\right) \right) + \frac{\pi^2 - 6}{6} + (1 - \gamma)^2 + 2(1 - \gamma) \ln\left(\frac{2\pi e^\gamma}{A^{12}}\right) + \frac{6\zeta''(2)}{\pi^2} \right)$$

$$I = \frac{\pi^2}{6\sigma^2} \left(\ln^2(\lambda) + 2\ln(\lambda) \left(\ln\left(\frac{2\pi e}{A^{12}}\right) \right) + \frac{\pi^2 - 6}{6} + (\gamma - 1)^2 - 2(\gamma - 1) \left(\gamma - 1 + \ln\left(\frac{2\pi e}{A^{12}}\right) \right) + \frac{6\zeta''(2)}{\pi^2} \right)$$

$$I = \frac{\pi^2}{6\sigma^2} \left(\ln^2(\lambda) + 2\ln(\lambda) \ln\left(\frac{2\pi e}{A^{12}}\right) + \frac{\pi^2 - 6}{6} - (\gamma - 1)^2 - 2(\gamma - 1) \ln\left(\frac{2\pi e}{A^{12}}\right) + \frac{6\zeta''(2)}{\pi^2} \right)$$

Completing the square and substituting $\frac{\alpha}{\sigma} = \lambda$

$$I = \frac{\pi^2}{6\sigma^2} \left(\ln^2\left(\frac{2\pi e\alpha}{A^{12}\sigma}\right) - \ln^2\left(\frac{2\pi e}{A^{12}}\right) + \frac{\pi^2 - 6}{6} - \left(\gamma - 1 + \ln\left(\frac{2\pi e}{A^{12}}\right) \right)^2 + \ln^2\left(\frac{2\pi e}{A^{12}}\right) + \frac{6\zeta''(2)}{\pi^2} \right)$$

$\therefore$

$$I = \frac{\pi^2}{6\sigma^2} \left(\ln^2\left(\frac{2\pi e\alpha}{A^{12}\sigma}\right) - \ln^2\left(\frac{2\pi e^\gamma}{A^{12}}\right) + \frac{\pi^2 - 6}{6} + \frac{6\zeta''(2)}{\pi^2} \right)$$

Where: γ = Euler – Mascheroni Constant, A = Glaisher – Kinkelin Constant