

ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$I = \int_0^{\infty} \frac{x e^{\cos(\alpha x)} \sin(\sin(\alpha x))}{x^2 + \gamma^2} dx$$

$$\forall \alpha > 0, \gamma > 0$$

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Solution by proposer

$$2I = \int_{-\infty}^{\infty} \frac{x e^{\cos(\alpha x)} \sin(\sin(\alpha x))}{x^2 + \gamma^2} dx \quad (\text{even function})$$

$$\text{Let } 2I = \Im(K): \quad \dots (0)$$

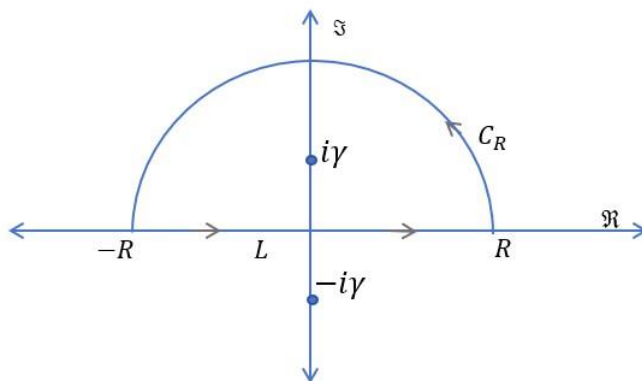
$$K = \int_{-\infty}^{\infty} \frac{x e^{\cos(\alpha x)} e^{i \sin(\alpha x)}}{x^2 + \gamma^2} dx = \int_{-\infty}^{\infty} \frac{x e^{\cos(\alpha x) + i \sin(\alpha x)}}{x^2 + \gamma^2} dx$$

$$K = \int_{-\infty}^{\infty} \frac{x e^{i \alpha x}}{x^2 + \gamma^2} dx$$

$$\text{Let } f(z) = \frac{z e^{i \alpha z}}{z^2 + \gamma^2}$$

With simple poles $z_0 = \pm i\gamma$

The figure below shows a semicircular contour 'C' with radius $R > \gamma$ enclosing $z_0 = i\gamma$



By Cauchy's Residue Theorem

$$\oint_C f(z) dz = \int_L f(z) dz + \int_{C_R} f(z) dz = 2\pi i (\text{Res}(f, z_0)) \quad \dots (1)$$

$$\text{Working on } \int_L f(z) dz: z = x e^{i0} = x; dz = dx$$

$$\int_L f(z) dz = \int_{-R}^R f(x) dx = \int_{-R}^R \frac{x e^{i \alpha x}}{x^2 + \gamma^2} dx$$

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$$\lim_{R \rightarrow \infty} \int_L f(z) dz = \lim_{R \rightarrow \infty} \int_{-R}^R \frac{x e^{i\alpha x}}{x^2 + \gamma^2} dx = \int_{-\infty}^{\infty} \frac{x e^{i\alpha x}}{x^2 + \gamma^2} dx \quad \dots (2)$$

Working on $\int_{C_R} f(z) dz$

$$f(z) = \frac{z e^{i\alpha z}}{z^2 + \gamma^2} = \frac{z}{z^2 + \gamma^2} \sum_{n=0}^{\infty} \frac{e^{i\alpha n z}}{n!} = \frac{z}{z^2 + \gamma^2} \left(1 + \sum_{n=1}^{\infty} \frac{e^{i\alpha n z}}{n!} \right) \quad (\text{Taylor Series})$$

$\therefore$

$$\begin{aligned} \int_{C_R} f(z) dz &= \int_{C_R} \frac{z}{z^2 + \gamma^2} dz + \int_{C_R} \frac{z}{z^2 + \gamma^2} \sum_{n=1}^{\infty} \frac{e^{i\alpha n z}}{n!} dz \\ \int_{C_R} f(z) dz &= \int_{C_R} \frac{z}{z^2 + \gamma^2} dz \\ &+ \sum_{n=1}^{\infty} \frac{1}{n!} \int_{C_R} \frac{z}{z^2 + \gamma^2} e^{i\alpha n z} dz \quad (\text{Uniform convergence}) \end{aligned}$$

$$z = R e^{i\theta}, dz = i R e^{i\theta} d\theta, \theta \in [0, \pi]$$

$$\int_{C_R} f(z) dz = i \int_0^{\pi} \frac{R e^{i\theta}}{R^2 e^{i2\theta} + \gamma^2} R e^{i\theta} d\theta + i \underbrace{\sum_{n=1}^{\infty} \frac{1}{n!} \int_0^{\pi} \frac{R e^{i\theta}}{R^2 e^{i2\theta} + \gamma^2} e^{i\alpha n z} R e^{i\theta} d\theta}_T$$

By Jordan's Lemma

$$\lim_{R \rightarrow \infty} \left| \int_0^{\pi} \frac{R e^{i\theta}}{R^2 e^{i2\theta} + \gamma^2} e^{i\alpha n z} R e^{i\theta} d\theta \right| \leq \lim_{R \rightarrow \infty} \frac{R}{R^2 - \gamma^2} \frac{\pi}{\alpha n} = 0 \quad \forall \alpha, n > 0$$

$$\therefore T = 0; R \rightarrow \infty$$

$\therefore$

$$\begin{aligned} \int_{C_R} f(z) dz &= i \int_0^{\pi} \frac{R e^{i\theta}}{R^2 e^{i2\theta} + \gamma^2} R e^{i\theta} d\theta = i \int_0^{\pi} \frac{R^2 e^{i2\theta}}{R^2 e^{i2\theta} + \gamma^2} d\theta \\ \lim_{R \rightarrow \infty} \int_{C_R} f(z) dz &= i \lim_{R \rightarrow \infty} \int_0^{\pi} \frac{R^2 e^{i2\theta}}{R^2 e^{i2\theta} + \gamma^2} d\theta = i \lim_{R \rightarrow \infty} \int_0^{\pi} \frac{1}{1 + \left(\frac{\gamma}{R e^{i\theta}}\right)^2} d\theta = i \int_0^{\pi} d\theta = i\pi \\ \lim_{R \rightarrow \infty} \int_{C_R} f(z) dz &= i\pi \quad \dots (3) \end{aligned}$$

Calculating Residue at $z_0 = i\gamma$

$$\text{Res}(f, z_0) = \lim_{z \rightarrow i\gamma} \frac{N(z)}{D'(z)} : D(z) = z^2 + \gamma^2, D'(z) = 2z$$

$$\text{Res}(f, z_0) = \lim_{z \rightarrow i\gamma} \frac{z e^{i\alpha z}}{2z} = \frac{1}{2} \lim_{z \rightarrow i\gamma} e^{i\alpha z} = \frac{1}{2} e^{i\alpha(i\gamma)} = \frac{1}{2} e^{-\alpha\gamma}$$

$$2\pi i \text{Res}(f, z_0) = 2\pi i \frac{1}{2} e^{-\alpha\gamma} = i\pi e^{-\alpha\gamma} \quad \dots (4)$$

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Taking $\lim_{R \rightarrow \infty}$ and substituting equations 2, 3 and 4 into equation 1, we get;

$$\lim_{R \rightarrow \infty} \oint_C f(z) dz = \lim_{R \rightarrow \infty} \int_L f(z) dz + \lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = \lim_{R \rightarrow \infty} 2\pi i (\text{Res}(f(z_0)))$$

$$\int_{-\infty}^{\infty} \frac{x e^{i\alpha x}}{x^2 + \gamma^2} dx + i\pi = i\pi e^{-\alpha\gamma}$$

$$K = \int_{-\infty}^{\infty} \frac{x e^{i\alpha x}}{x^2 + \gamma^2} dx = i\pi e^{-\alpha\gamma} - i\pi = i\pi(e^{-\alpha\gamma} - 1)$$

From equation 0

$$2I = \Im(K)$$

$$\therefore 2I = \pi(e^{-\alpha\gamma} - 1)$$

$$I(\alpha, \gamma) = \frac{\pi}{2}(e^{-\alpha\gamma} - 1) \quad \forall \alpha, \gamma > 0$$

$$\text{Example: } I(1, 1) = \frac{\pi}{2}(e^{e^{-(1)(1)}} - 1) = \frac{\pi}{2}(e^{e^{-1}} - 1) \approx 0.698483$$