

ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$I(\alpha, \gamma) = \int_0^{\infty} \frac{\cos(\alpha x) - \sin(\alpha x) + \sqrt{2} \cos(\alpha x)}{\sqrt[4]{x} (x^2 + \gamma^2)^2} dx$$

$\forall \alpha > 0, \gamma > 0$

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Solution by proposer

Consider $f(z) = \frac{e^{iaz}}{z^{\beta}(z^2 + \gamma^2)^2}$

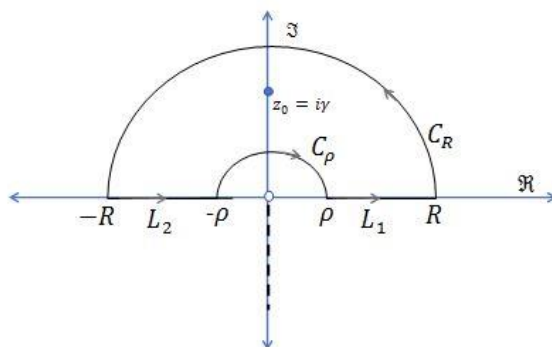
Notice: The branch point $z = 0$ for $\beta \notin \mathbb{Z}$

Hence, we place a branch cut along the $-\Im$ axis: $-\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$

$\therefore e^{i\pi} = -1, e^{i\frac{\pi}{2}} = i$ are valid

$f(z)$ has an order 2 pole at $z_0 = i\gamma$ (above the branch cut)

The region shown below shows an indented semicircular contour 'C' on which $f(z)$ is meromorphic



By Cauchy's Residue Theorem:

$$\oint_C f(z) dz = \int_{L_1} f(z) dz + \int_{C_R} f(z) dz + \int_{L_2} f(z) dz + \int_{C_\rho} f(z) dz$$

$$= 2\pi i (\text{Res}(f(z_0))) \dots (1)$$

Working on L_1 & $L_2, z = re^{i\theta}$:

$L_1: z = re^{i0}: x, dz = dx$

$L_2: z = re^{i\pi}: -x, dz = -dx$

$\therefore$

$$\int_{L_1} f(z) dz + \int_{L_2} f(z) dz = \int_{\rho}^R \frac{e^{i\alpha x}}{x^{\beta}(x^2 + \gamma^2)^2} dx - \int_R^{\rho} \frac{e^{-i\alpha x}}{(-1)^{\beta} x^{\beta}(x^2 + \gamma^2)^2} dx$$

$$\int_{L_1} f(z) dz + \int_{L_2} f(z) dz = \int_{\rho}^R \frac{e^{i\alpha x}}{x^{\beta}(x^2 + \gamma^2)^2} dx + \int_{\rho}^R \frac{e^{-i\alpha x}}{e^{i\pi\beta} x^{\beta}(x^2 + \gamma^2)^2} dx$$

$$\int_{L_1} f(z) dz + \int_{L_2} f(z) dz = \int_{\rho}^R \frac{e^{i\alpha x} + e^{-i\alpha x} e^{-i\pi\beta}}{x^{\beta}(x^2 + \gamma^2)^2} dx$$

$$\text{Taking } \lim_{\substack{\rho \rightarrow 0 \\ R \rightarrow \infty}} \int_{\rho}^R \frac{e^{i\alpha x} + e^{-i\alpha x} e^{-i\pi\beta}}{x^{\beta}(x^2 + \gamma^2)^2} dx$$

$$\int_{L_1} f(z) dz + \int_{L_2} f(z) dz = \int_0^{\infty} \frac{e^{i\alpha x} + e^{-i\alpha x} e^{-i\pi\beta}}{x^{\beta}(x^2 + \gamma^2)^2} dx \dots (2)$$

$$\text{Working on } \int_{C_R} f(z) dz \text{ \& } \int_{C_{\rho}} f(z) dz$$

$$\text{Let } f(z) = g(z) e^{i\alpha z}, \alpha > 0$$

$$\therefore \int_{C_R} f(z) dz = 0 \text{ by Jordan's Lemma } \dots (3)$$

$$\text{For } C_{\rho}: z = a + ib, \quad |z| = \rho$$

$$\left| \int_{C_{\rho}} f(z) dz \right| \leq \frac{1}{\rho^{\beta}(\rho^2 - \gamma^2)^2} \pi \rho \text{ (ML inequality)}$$

$$\lim_{\rho \rightarrow 0} \frac{1}{\rho^{\beta}(\rho^2 - \gamma^2)^2} \pi \rho = 0 : \beta < 1$$

$$\therefore \int_{C_{\rho}} f(z) dz = 0 \dots (4)$$

Now calculating the residue at z_0 (order 2 pole)

$$\text{Res}(f(z_0)) = \lim_{z \rightarrow i\gamma} \phi'(z) : \phi(z) = \frac{e^{i\alpha z}}{z^{\beta}(z + i\gamma)^2}$$

$$\phi'(z) = \frac{e^{i\alpha z}}{z^{\beta}(z + i\gamma)^2} \left(i\alpha - \frac{\beta}{z} - \frac{2}{z + i\gamma} \right)$$

$$\text{Res}(f(z_0)) = \lim_{z \rightarrow i\gamma} \frac{e^{i\alpha z}}{z^{\beta}(z + i\gamma)^2} \left(i\alpha - \frac{\beta}{z} - \frac{2}{z + i\gamma} \right) = \frac{e^{-\alpha\gamma}}{-4\gamma^2 e^{i\beta\frac{\pi}{2}} \gamma^{\beta}} \left(i\alpha - \frac{\beta}{i\gamma} - \frac{1}{i\gamma} \right)$$

$$\text{Res}(f(z_0)) = i \frac{e^{-\alpha\gamma}}{-4 \gamma^{\beta+3}} (\alpha\gamma + \beta + 1) e^{-i\beta\frac{\pi}{2}}$$

$$\text{Res}(f(z_0)) = i \frac{e^{-\alpha\gamma}}{-4 \gamma^{\beta+3}} (\alpha\gamma + \beta + 1) \left(\cos\left(\beta\frac{\pi}{2}\right) - i \sin\left(\beta\frac{\pi}{2}\right) \right)$$

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$$\therefore 2\pi i(\text{Res}(f(z_0))) = \frac{\pi e^{-\alpha\gamma}}{2\gamma^{\beta+3}}(\alpha\gamma + \beta + 1) \left(\cos\left(\beta\frac{\pi}{2}\right) - i \sin\left(\beta\frac{\pi}{2}\right) \right) \dots (5)$$

Substituting equation 2 to 5 in equation 1 and $\beta = \frac{1}{4}$:

$$\int_0^{\infty} \frac{e^{i\alpha x} + e^{-i\alpha x} e^{-i\frac{\pi}{4}}}{x^{\frac{1}{4}}(x^2 + \gamma^2)^2} dx = \frac{\pi e^{-\alpha\gamma}}{2\gamma^{\frac{13}{4}}} \left(\alpha\gamma + \frac{5}{4} \right) \left(\cos\left(\frac{\pi}{8}\right) - i \sin\left(\frac{\pi}{8}\right) \right)$$

$$\text{Since: } e^{-i\frac{\pi}{4}} = \cos\left(\frac{\pi}{4}\right) - i \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}(1 - i)$$

$$\Re \left\{ \int_0^{\infty} \frac{\cos(\alpha x) + i \sin(\alpha x) + \frac{1}{\sqrt{2}}(1 - i) (\cos(\alpha x) - i \sin(\alpha x))}{\sqrt[4]{x}(x^2 + \gamma^2)^2} dx \right\}$$

$$= \Re \left\{ \frac{\pi e^{-\alpha\gamma}}{8\sqrt[4]{\gamma^{13}}} (4\alpha\gamma + 5) \left(\cos\left(\frac{\pi}{8}\right) - i \sin\left(\frac{\pi}{8}\right) \right) \right\}$$

$$\int_0^{\infty} \frac{\cos(\alpha x) + \frac{1}{\sqrt{2}}(\cos(\alpha x) - \sin(\alpha x))}{\sqrt[4]{x}(x^2 + \gamma^2)^2} dx = \frac{\pi e^{-\alpha\gamma}}{8\sqrt[4]{\gamma^{13}}} (4\alpha\gamma + 5) \left(\cos\left(\frac{\pi}{8}\right) \right)$$

$$I(\alpha, \gamma) = \int_0^{\infty} \frac{\cos(\alpha x) - \sin(\alpha x) + \sqrt{2} \cos(\alpha x)}{\sqrt[4]{x}(x^2 + \gamma^2)^2} dx = \frac{\pi e^{-\alpha\gamma}}{4\sqrt{2} \sqrt[4]{\gamma^{13}}} (4\alpha\gamma + 5) \left(\cos\left(\frac{\pi}{8}\right) \right)$$

$$\text{Since: } \cos\left(\frac{\pi}{8}\right) = \frac{\sqrt{2 + \sqrt{2}}}{2}$$

$$\therefore$$

$$I(\alpha, \gamma) = \int_0^{\infty} \frac{\cos(\alpha x) - \sin(\alpha x) + \sqrt{2} \cos(\alpha x)}{\sqrt[4]{x}(x^2 + \gamma^2)^2} dx$$

$$= \frac{\pi e^{-\alpha\gamma}}{8\sqrt{2} \sqrt[4]{\gamma^{13}}} (4\alpha\gamma + 5) \left(\sqrt{2 + \sqrt{2}} \right)$$

$$\forall \alpha > 0, \gamma > 0$$