ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\int_0^1 \ln^3(x)\left(\frac{\ln(1-x)}{1-x} - Li_2(x^3) - \frac{\ln(1+x)}{1+x}\right)dx =$$

$$= \frac{105}{16}\zeta(5) - \frac{\pi^2}{2}\zeta(3) + 18\zeta\left(2,\frac{1}{3}\right) + 4\zeta\left(3,\frac{1}{3}\right) + \frac{2}{3}\zeta\left(4,\frac{1}{3}\right) + \pi^2 + 12\pi\sqrt{3} + 108\ln(3) - 540$$

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$$I = \underbrace{\int_0^1 \frac{\ln^3(x)\ln(1-x)}{1-x}dx}_{I_1} - \underbrace{\int_0^1 \ln^3(x) Li_2(x^3)dx}_{I_2} - \underbrace{\int_0^1 \frac{\ln^3(x)\ln(1+x)}{1+x}dx}_{I_3}$$

Generating function for the harmonic numbers

Note: $\frac{\ln(1-x)}{1-x} = -\sum_{n=1}^{\infty} x^n H_n, \frac{\ln(1+x)}{(1+x)} = -\sum_{n=1}^{\infty}(-1)^n x^n H_n$

$$I_1 = \int_0^1 \frac{\ln^3(x)\ln(1-x)}{1-x}dx = -\sum_{n=1}^{\infty} H_n \int_0^1 x^n \ln^3(x)\,dx$$

Note: $2\int_0^1 x^n \ln^m(x)dx = \frac{(-1)^m\Gamma(m+1)}{(n+1)^{m+1}}, n > -1\ m \in Z^+$

$$I_1 = 6\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^4} = 6\sum_{n=1}^{\infty}\frac{H_{n+1} - \frac{1}{n+1}}{(n+1)^4} = 6\sum_{n=1}^{\infty}\frac{H_{n+1}}{(n+1)^4} - 6\sum_{n=1}^{\infty}\frac{1}{(n+1)^5} =$$

$$= 6\sum_{n=1}^{\infty}\frac{H_n}{n^4} - 6 - 6\left(\sum_{n=1}^{\infty}\frac{1}{n^5} - 1\right) = 6\sum_{n=1}^{\infty}\frac{H_n}{n^4} - 6\zeta(5)$$

Let q ∈ Z>1. Then the following identity holds:

$$\sum_{n=1}^{\infty}\frac{H_n}{n^q} = \frac{1}{2}(q+2)\zeta(q+1) - \frac{1}{2}\sum_{k=1}^{q-2}\zeta(q-k)\zeta(k+1)$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{n^{2q}} = -\frac{1}{2}(2q+1)\eta(2q+1) + \sum_{k=0}^{q-1}\zeta(2q-2k+1)\eta(2k)$$

- $I_1 = 18\zeta(5) - 6\zeta(2)\zeta(3) - 6\zeta(5) = 12\zeta(5) - \pi^2\zeta(3)$

$$I_2 = \int_0^1 \ln^3(x) Li_2(x^3)dx = \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^1 x^{3n} \ln^3(x)dx = -6\sum_{n=1}^{\infty} \frac{1}{n^2(3n+1)^4} =$$

$$= -6\sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{12}{n(3n+1)} + \frac{27}{(3n+1)^2} + \frac{18}{(3n+1)^3} + \frac{9}{(3n+1)^4}\right)$$

Note: Hurwitz zeta function.

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(a+n)^s}, \zeta(s,a) = \frac{1}{a^s} + \zeta(s,a+1), \ \ Re\{s\} > 1, \qquad a \neq (0,-1,-2,\dots)$$

$$I_2 = -6\left(\zeta(2) - 12\sum_{n=1}^{\infty} \frac{1}{n(3n+1)} + \sum_{n=0}^{\infty} \frac{27}{(3n+4)^2} + \sum_{n=0}^{\infty} \frac{18}{(3n+4)^3} + \sum_{n=0}^{\infty} \frac{9}{(3n+4)^4}\right) =$$

$$= -\pi^2 + 24\sum_{n=1}^{\infty} \frac{1}{n\left(n+\frac{1}{3}\right)} - 18\zeta\left(2,\frac{4}{3}\right) - 4\zeta\left(3,\frac{4}{3}\right) - \frac{2}{3}\zeta\left(4,\frac{4}{3}\right) =$$

$$= -\pi^2 - 18\zeta\left(2,\frac{1}{3}\right) + 162 - 4\zeta\left(3,\frac{1}{3}\right) + 108 - \frac{2}{3}\zeta\left(4,\frac{1}{3}\right) + 54 + 24\sum_{n=1}^{\infty} \frac{1}{n\left(n+\frac{1}{3}\right)} =$$

$$= -\pi^2 - 18\zeta\left(2,\frac{1}{3}\right) - 4\zeta\left(3,\frac{1}{3}\right) - \frac{2}{3}\zeta\left(4,\frac{1}{3}\right) + 24J + 324$$

$$J = \sum_{n=1}^{\infty} \frac{1}{n\left(n+\frac{1}{3}\right)} = \sum_{n=0}^{\infty} \frac{1}{(n+1)\left(n+\frac{4}{3}\right)} = \sum_{n=0}^{\infty} \int_0^1 \int_0^1 x^n y^{n+\frac{1}{3}} dxdy =$$

$$= \int_0^1 \int_0^1 y^{\frac{1}{3}} \sum_{n=0}^{\infty} (xy)^n \, dxdy = \int_0^1 y^{\frac{1}{3}} \int_0^1 \frac{1}{1-xy} dxdy = \int_0^1 y^{\frac{1}{3}} \left[\frac{\ln(1-xy)}{y}\right]_0^1 dy =$$

$$= -\int_0^1 y^{-\frac{2}{3}} \ln(1-y) \, dy \ \rightarrow \ J(a) = \int_0^1 y^{-\frac{2}{3}} (1-y)^a \, dy \ \rightarrow \ J = -\lim_{a\to 0} \frac{d}{da} J(a)$$

$$J(a) = \int_0^1 y^{\frac{1}{3}-1} (1-y)^{a+1-1} dy = \beta\left(\frac{1}{3}, a+1\right) = \frac{\Gamma\left(\frac{1}{3}\right)\Gamma(a+1)}{\Gamma\left(\frac{4}{3}+a\right)}$$

$$J=-\Gamma\left(\frac{1}{3}\right)\lim_{a\to 0}\frac{d}{da}\frac{\Gamma(a+1)}{\Gamma\left(\frac{4}{3}+a\right)}=-\Gamma\left(\frac{1}{3}\right)\lim_{a\to 0}\frac{\Gamma(a+1)(\psi^{(0)}(a+1)-\psi^{(0)}\left(a+\frac{4}{3}\right)}{\Gamma\left(\frac{4}{3}+a\right)}=$$

$$=-\frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{4}{3}\right)}\left(\psi^{(0)}(1)-\psi^{(0)}\left(\frac{4}{3}\right)\right)=-3\left(-\gamma-3+\gamma+\frac{\pi}{2\sqrt{3}}+\frac{3}{2}\ln(3)\right)=9-\frac{3\pi}{2\sqrt{3}}-\frac{9}{2}\ln(3)$$

$$I_2=-\pi^2-18\zeta\left(2,\frac{1}{3}\right)-4\zeta\left(3,\frac{1}{3}\right)-\frac{2}{3}\zeta\left(4,\frac{1}{3}\right)+324+24\left(9-\frac{\pi\sqrt{3}}{2}-\frac{9}{2}\ln(3)\right)$$

$$I_3=\int_0^1\frac{\ln^3(x)\ln(1+x)}{1+x}dx=-\sum_{n=1}^{\infty}(-1)^n H_n\int_0^1 x^n\ln^3(x)dx=$$

$$=6\sum_{n=1}^{\infty}\frac{(-1)^n H_n}{(n+1)^4}=6\sum_{n=1}^{\infty}\frac{(-1)^n H_{n+1}}{(n+1)^4}-6\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+1)^5}=$$

$$=6\sum_{n=1}^{\infty}\frac{(-1)^{n+1}H_n}{n^4}-6\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^5}=-6\left(-\frac{5}{2}\eta(5)+\sum_{k=0}^{1}\zeta(5-2k)\,\eta(2k)\right)-\frac{45}{8}\zeta(5)$$

$$=15\eta(5)-3\zeta(5)-6\zeta(3)\eta(2)-\frac{45}{8}\zeta(5)$$

$$=\frac{225}{16}\zeta(5)-3\zeta(5)-\frac{45}{8}\zeta(5)-3\zeta(3)\zeta(2)=\frac{87}{16}\zeta(5)-\frac{\pi^2}{2}\zeta(3)$$

$$I=I_1-I_2-I_3=$$

$$=12\zeta(5)-\pi^2\zeta(3)+\pi^2+18\zeta\left(2,\frac{1}{3}\right)+4\zeta\left(3,\frac{1}{3}\right)+\frac{2}{3}\zeta\left(4,\frac{1}{3}\right)-540+12\pi\sqrt{3}$$
$$+108\ln(3)-\frac{87}{16}\zeta(5)+\frac{\pi^2}{2}\zeta(3)=$$

$$=\frac{105}{16}\zeta(5)-\frac{\pi^2}{2}\zeta(3)+\pi^2+18\zeta\left(2,\frac{1}{3}\right)+4\zeta\left(3,\frac{1}{3}\right)+\frac{2}{3}\zeta\left(4,\frac{1}{3}\right)-540+12\pi\sqrt{3}$$
$$+108\ln(3)$$

Note: Dirichlet eta function, $\eta(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}=(1-2^{1-s})\zeta(s)\quad Re\{s\}>0$

$$\eta(0)=\frac{1}{2}$$