ROMANIAN MATHEMATICAL MAGAZINE

Solve for reals:

$$\int_0^1\left(\sqrt[9]{\left(x^{9x^4+9}+x^{9x^2}+x^{18}\right)}\right)^5 dx=\frac{\sqrt[9]{4782969}}{5x^4+10x^2+18}$$

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$$\text{We will show}: a,b,c>0, \int_0^1\left(\sqrt[9]{\left(x^{9a}+x^{9b}+x^{9c}\right)}\right)^5 dx\ge\frac{\sqrt[9]{4782969}}{5(a+b+c)+3}$$

we know that ,

if $a_1,a_2,\dots,a_n$ *be n positive real numbers and m be a rational number then,* $\frac{a_1^m+a_2^m+\dots+a_n^m}{n}>\text{ or }<\left(\frac{a_1+a_2+\dots+a_n}{n}\right)^m$

according as m does not or does lie between 0 *and* 1

(*Reference: S.K. Mapa classical algebra book , page* -22)

$$\frac{\left((x^{9a})^{\frac{5}{9}}+(x^{9b})^{\frac{5}{9}}+(x^{9c})^{\frac{5}{9}}\right)}{3}\le\left(\frac{x^{9a}+x^{9b}+x^{9c}}{3}\right)^{\frac{5}{9}}\ as\ \frac{5}{9}\in(0,1)$$

$$\left(x^{9a}+x^{9b}+x^{9c}\right)^{\frac{5}{9}}\ge\frac{1}{3^{\frac{4}{9}}}\left((x^{9a})^{\frac{5}{9}}+(x^{9b})^{\frac{5}{9}}+(x^{9c})^{\frac{5}{9}}\right)$$

$$\left(\sqrt[9]{x^{9a}+x^{9b}+x^{9c}}\right)^5\ge\frac{1}{3^{\frac{4}{9}}}\left(x^{5a}+x^{5b}+x^{5c}\right)\ (1)$$

$$\int_0^1\left(\sqrt[9]{\left(x^{9a}+x^{9b}+x^{9c}\right)}\right)^5 dx\overset{(1)}{\ge}\frac{1}{3^{\frac{4}{9}}}\int_0^1\left(x^{5a}+x^{5b}+x^{5c}\right)dx=$$

$$=\frac{1}{3^{\frac{4}{9}}}\left[\left(\frac{x^{5a+1}}{5a+1}\right)_0^1+\left(\frac{x^{5b+1}}{5b+1}\right)_0^1+\left(\frac{x^{5c+1}}{5c+1}\right)_0^1\right]=$$

$$\ge\frac{1}{3^{\frac{4}{9}}}\sum\frac{1}{5a+1}\overset{Bergstrom}{\ge}\frac{1}{3^{\frac{4}{9}}}\frac{9}{5(a+b+c)+3}=$$

$$=\frac{3^{\frac{14}{5}}}{5(a+b+c)+3}\ge\frac{\sqrt[9]{3^{14}}}{5(a+b+c)+3}=\frac{\sqrt[9]{4782969}}{5(a+b+c)+3}$$

Equality holds for $a=b=c$

now if we take $a=x^4+1, b=2x^2, c=2$ *then*

$$\int_0^1\left(\sqrt[9]{\left(x^{9x^4+9}+x^{9x^2}+x^{18}\right)}\right)^5 dt\ge\frac{\sqrt[9]{4782969}}{5(x^4+1+2x^2+2)+3}=\frac{\sqrt[9]{4782969}}{5x^4+10x^2+18}$$

Equlity occurs when $a=b=c$ *or,* $x^4+1=2x^2=2$ *or,* $x^2=1$ *or,* $x=\pm1$