

ROMANIAN MATHEMATICAL MAGAZINE

Evaluate the closed form value of the integral:

$$I = \int_0^{\infty} x^{\frac{1}{2}} e^{-3x} \ln x \cos 2x \, dx$$

where γ is the Euler – Mascheroni constant

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Solution by proposer

$$I = \Re \int_0^{\infty} x^{\frac{1}{2}} e^{-(3-2i)x} \ln x \, dx \Rightarrow \int_0^{\infty} y^k e^{-ay} \ln y \, dy = \frac{\Gamma(k+1)}{a^{k+1}} \stackrel{\text{Differentiate wrt } k}{\Rightarrow} \\ \Rightarrow \int_0^{\infty} y^k e^{-ay} \ln y \, dy = \frac{(a^{k+1} \Gamma'(k+1) - \Gamma(k+1) a^{k+1} \ln a)}{a^{2k+2}}$$

$$I = \Re \int_0^{\infty} x^{\frac{1}{2}} e^{-(3-2i)x} \ln x \, dx = \Re \left(\int_0^{\infty} y^k e^{-ay} \ln y \, dy \right) \text{ at } k = \frac{1}{2}, a = 3 - 2i$$

$$I = \Re \left(\frac{(a^{k+1} \Gamma'(k+1) - \Gamma(k+1) a^{k+1} \ln a)}{a^{2k+2}} \right)$$

$$= \Re(\psi(k+1) \Gamma(k+1) a^{-k-1} - \Gamma(k+1) a^{-k-1} \ln a)$$

$$= \Re \left(\Gamma(k+1) a^{-k-1} (\psi(k+1) - \ln a) \right) = \Gamma\left(\frac{3}{2}\right) \Re \left((3-2i)^{-\frac{3}{2}} \left(\psi\left(\frac{3}{2}\right) - \ln(3-2i) \right) \right)$$

$$= \frac{\sqrt{\pi}}{2} \Re \left((3-2i)^{-\frac{3}{2}} (2 - \gamma - 2 \ln 2 - \ln(3-2i)) \right)$$

$$\ln(3-2i) = \frac{1}{2} \ln 13 - i \tan^{-1} \frac{2}{3}$$

$$(3-2i)^{-\frac{3}{2}} = 13^{-\frac{3}{4}} \left(e^{-\frac{3}{2}i} \right)^{i\theta} = 13^{-\frac{3}{4}} \left(\cos\left(\frac{3}{2} \tan^{-1} \frac{2}{3}\right) + i \sin\left(\frac{3}{2} \tan^{-1} \frac{2}{3}\right) \right)$$

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$$I = \frac{\sqrt{\pi}}{2} \Re \left(13^{-\frac{3}{4}} \left(\cos \left(\frac{3}{2} \tan^{-1} \frac{2}{3} \right) + i \sin \left(\frac{3}{2} \tan^{-1} \frac{2}{3} \right) \right) \left(2 - \gamma - 2 \ln 2 - \left(\frac{1}{2} \ln 13 - i \tan^{-1} \frac{2}{3} \right) \right) \right)$$

$$I = \int_0^{\infty} x^{\frac{1}{2}} e^{-3x} \ln x \cos 2x \, dx$$

$$= \frac{\sqrt{\pi}}{2} 13^{-\frac{3}{4}} \left[\cos \left(\frac{3}{2} \tan^{-1} \frac{2}{3} \right) \left(2 - \gamma - 2 \ln 2 - \frac{1}{2} \ln 13 \right) - \sin \left(\frac{3}{2} \tan^{-1} \frac{2}{3} \right) \tan^{-1} \frac{2}{3} \right]$$