

ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\int_0^{\infty} e^{-(\ln x)^{2a}} dx = \frac{1}{a} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{n}{a} + \frac{1}{2a}\right)}{\Gamma(2n+1)}$$

Proposed by Omkumar Rao-India

Solution by proposer

$$\begin{aligned} \int_0^{\infty} e^{-(\ln x)^{2a}} dx &\stackrel{\ln x \rightarrow x}{\Rightarrow} \int_{-\infty}^{\infty} e^{-(x)^{2a}} e^x dx = \int_0^{\infty} e^{-(x)^{2a}} (e^x + e^{-x}) dx = \\ &= 2 \int_0^{\infty} e^{-(x)^{2a}} \cosh(x) dx = 2 \sum_{n=0}^{\infty} \frac{1}{\Gamma(2n+1)} \int_0^{\infty} x^{2n} e^{-(x)^{2a}} dx \\ \int_0^{\infty} x^{2n} e^{-(x)^{2a}} dx &\stackrel{x^{2a} \rightarrow x}{\Rightarrow} \frac{1}{2a} \int_0^{\infty} x^{\left(\frac{n}{a} + \frac{1}{2a}\right)-1} e^{-x} dx = \frac{1}{2a} \Gamma\left(\frac{n}{a} + \frac{1}{2a}\right) \end{aligned}$$

$$\int_0^{\infty} e^{-(\ln x)^{2a}} dx = \frac{1}{a} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{n}{a} + \frac{1}{2a}\right)}{\Gamma(2n+1)}$$