

ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\int_0^{\frac{\pi}{2}} \frac{\ln(\cos(x))}{x^2 + (\ln(\cos(x)))^2} dx = \frac{\pi}{2} (1 - \log_2(e))$$

Proposed by Omkumar Rao-India

Solution by proposer

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{\ln(\cos(x))}{x^2 + (\ln(\cos(x)))^2} dx &= \int_0^{\frac{\pi}{2}} \frac{1}{1 + \left(\frac{x}{\ln(\cos(x))}\right)^2} dx = \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{1 + \left(\frac{x}{\ln(\cos(x))}\right)^2} + \frac{x \tan(x)}{(\ln(\cos(x)))^2} - \frac{x \tan(x)}{(\ln(\cos(x)))^2} dx = \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{1 + \left(\frac{x}{\ln(\cos(x))}\right)^2} dx - \int_0^{\frac{\pi}{2}} \frac{x \tan(x)}{1 + \left(\frac{x}{\ln(\cos(x))}\right)^2} dx = \\ &= \int_0^{\frac{\pi}{2}} \frac{d\left(\frac{x}{\ln(\cos(x))}\right)}{1 + \left(\frac{x}{\ln(\cos(x))}\right)^2} dx - \underbrace{\int_0^{\frac{\pi}{2}} \frac{x \tan(x)}{x^2 + (\ln(\cos(x)))^2} dx}_{-\ln(\cos(x)) \rightarrow x} = \\ &= \left[\tan^{-1}\left(\frac{x}{\ln(\cos(x))}\right) \right]_0^{\frac{\pi}{2}} - \int_0^{\infty} \frac{\cos^{-1}(e^{-x})}{x^2 + \cos^{-1}(e^{-x})} dx = \\ &= \frac{\pi}{2} - \int_0^{\infty} \left(\underbrace{\int_0^{\infty} e^{-yx} \sin(y \cos^{-1}(e^{-x})) dy}_{e^{-x} \rightarrow x} \right) dx = \end{aligned}$$

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$$\begin{aligned}
 &= \frac{\pi}{2} - \int_0^{\infty} \left(\underbrace{\int_0^1 x^{y-1} \sin(y(\cos^{-1}(x))) dy}_{\text{Now by parts}} \right) dx = \\
 &= \frac{\pi}{2} - \int_0^{\infty} \left(\underbrace{\int_0^1 x^y \cos(y(\cos^{-1}(x))) dx}_{x \rightarrow \cos x} \right) dy = \\
 &= \frac{\pi}{2} - \int_0^{\infty} \left(\int_0^{\frac{\pi}{2}} (\cos(x))^y \cos(yx) dx \right) dy = \frac{\pi}{2} - \int_0^{\infty} \Re \left(\int_0^{\frac{\pi}{2}} (\cos(x))^y e^{iyx} dx \right) dy \\
 &= \frac{\pi}{2} - \int_0^{\infty} \Re \left(\int_0^{\frac{\pi}{2}} ((\cos(x))^2 + i \sin(x) \cos(x))^y dx \right) dy \\
 &= \frac{\pi}{2} - \int_0^{\infty} \Re \left(\frac{1}{2^y} \int_0^{\frac{\pi}{2}} (1 + e^{2ix})^y dx \right) dy \\
 &= \frac{\pi}{2} - \int_0^{\infty} \frac{1}{2^y} \Re \left(\int_0^{\frac{\pi}{2}} \left(\sum_{j=0}^y e^{i2jx} \right) dx \right) dy = \frac{\pi}{2} - \int_0^{\infty} \frac{1}{2^y} \Re \left(\sum_{j=0}^y \left(\int_0^{\frac{\pi}{2}} e^{i2jx} dx \right) \right) dy = \\
 &= \frac{\pi}{2} - \int_0^{\infty} \frac{\pi}{2^{y+1}} dy = \frac{\pi}{2} - \frac{\pi}{2 \ln(2)} = \frac{\pi}{2} (1 - \log_2(e))
 \end{aligned}$$

$$\int_0^{\frac{\pi}{2}} \frac{\ln(\cos(x))}{x^2 + (\ln(\cos(x)))^2} dx = \frac{\pi}{2} (1 - \log_2(e))$$