

ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$K = \int_0^{\infty} \frac{(\ln x)^2 \sqrt{x}}{(x^2 + 1)(x + 1)} dx$$

Proposed by Omkumar Rao-India

Solution by proposer

$$\begin{aligned} \int_0^{\infty} \frac{(\ln x)^2 \sqrt{x}}{(x^2 + 1)(x + 1)} dx &\stackrel{x \rightarrow x^2}{=} 8 \int_0^{\infty} \frac{(\ln x)^2 x^2}{(x^4 + 1)(x^2 + 1)} dx \\ &= 8 \left(\int_0^1 \frac{(\ln x)^2 x^2}{(x^4 + 1)(x^2 + 1)} dx + \underbrace{\int_1^{\infty} \frac{(\ln x)^2 x^2}{(x^4 + 1)(x^2 + 1)} dx}_{x \rightarrow \frac{1}{x}} \right) \\ K &= 16 \int_0^1 \frac{(\ln x)^2 x^2}{(x^4 + 1)(x^2 + 1)} dx = 8 \int_0^1 \frac{x^2 + 1}{x^4 + 1} (\ln x)^2 dx - 8 \int_0^1 \frac{(\ln x)^2}{x^2 + 1} dx \\ \int_0^1 \frac{x^2 + 1}{x^4 + 1} (\ln x)^2 dx &= \underbrace{\int_0^1 \frac{(\ln x)^2 x^2}{x^4 + 1} dx}_{x \rightarrow \frac{1}{x}} + \int_0^1 \frac{(\ln x)^2}{x^4 + 1} dx = \int_0^{\infty} \frac{(\ln x)^2}{x^4 + 1} dx = \frac{\partial^2 (P(m, n))}{\partial m^2} \\ P(m, n) &= \int_0^{\infty} \frac{x^m}{x^n + 1} = \frac{\pi}{n \sin \left(\frac{\pi(m+1)}{n} \right)} \\ K &= 8 \left(P''(0, 4) - \frac{\pi^3}{16} \right) = 8 \left(\frac{3\sqrt{2}\pi^3}{64} - \frac{\pi^3}{16} \right) = \frac{\pi^3}{8} (3\sqrt{2} - 4) \\ K &= \int_0^{\infty} \frac{(\ln x)^2 \sqrt{x}}{(x^2 + 1)(x + 1)} dx = \frac{\pi^3}{8} (3\sqrt{2} - 4) \end{aligned}$$