ROMANIAN MATHEMATICAL MAGAZINE

Find a closed form:

$$\Omega = \int_0^\infty \frac{x \ln x}{\sinh x} dx$$

Proposed by Omkumar Rao-India

Solution by proposer

$$\overset{\sinh x \to \frac{e^x - e^{-x}}{2}}{\Longrightarrow} \int_0^\infty x \ln x \left(2 \sum_{r=0}^\infty e^{-(2r+1)x}\right) dx = 2 \sum_{r=0}^\infty \int_0^\infty x \ln x \, e^{-(2r+1)x} dx$$

$$I(s) = \int_0^\infty x^s e^{-ax} dx = \frac{\Gamma(s+1)}{a^{s+1}}$$

$$I'(s) = \int_0^\infty x^s \ln x \, e^{-ax} dx = \frac{(a^{s+1}\Gamma'(s+1) - \Gamma(s+1)a^{s+1} \ln a)}{a^{2s+2}}$$

$$I'(1) = \frac{(a^2\Gamma'(2) - \Gamma(2)a^2 \ln a)}{a^4} = \frac{(\Gamma'(2) - \Gamma(2) \ln a)}{a^2}$$

$$\Gamma'(2) = \psi(2)\Gamma(2)$$

$$I'(1) = \frac{\Gamma(2)(\psi(2) - \ln a)}{a^2} : (a = 2r+1)$$

$$\int_0^\infty x \ln x \, e^{-(2r+1)x} dx = \frac{(1 - \gamma - \ln((2r+1)))}{(2r+1)^2}$$

$$\Omega = 2 \sum_{r=0}^\infty \left(\frac{1-\gamma}{(2r+1)^2} - \frac{\ln((2r+1))}{(2r+1)^2}\right) = 2\frac{\pi^2}{8}(1-\gamma) - 2 \sum_{r=0}^\infty \frac{\ln((2r+1))}{(2r+1)^2}$$

$$= \frac{\pi^2(1-\gamma)}{4} - 2 \sum_{r=0}^\infty \frac{\ln((2r+1))}{(2r+1)^2}$$

$$\textit{Dirichlet Lambda Function}: \ \lambda(s) = \sum_{n=0}^\infty \frac{1}{(2n+1)^s}$$

$$\lambda'(s) = -\sum_{n=0}^\infty \frac{\ln((2n+1))}{(2n+1)^s}$$

$$\lambda(s) = (1 - 2^{-s})\zeta(s)$$

$$\lambda'(s) = 2^{-s} \ln 2 \, \zeta(s) + (1 - 2^{-s})\zeta'(s) \overset{\lambda'(2)}{\Longrightarrow} 2^{-2} \ln 2 \, \zeta(2) + (1 - 2^{-2})\zeta'(2)$$

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s) \Longrightarrow \ln \zeta(s)$$

$$= s \ln 2 + (s-1) \ln \pi + \ln\left(\sin\left(\frac{\pi s}{2}\right)\right) + \ln \Gamma(1-s) + \ln \zeta(1-s)$$

$$\ln\zeta(s)=s\ln 2+(s-1)\ln\pi+\ln\left(\sin\left(\frac{\pi s}{2}\right)\right)+\ln\Gamma(1-s)+\ln\zeta(1-s)\Rightarrow\frac{\zeta'(s)}{\zeta(s)}$$

$$=\ln 2+\ln\pi+\frac{\pi}{2}\cot\left(\frac{\pi s}{2}\right)-\psi(1-s)-\frac{\zeta'(1-s)}{\zeta(1-s)}$$

$$\frac{\zeta'(s)}{\zeta(s)}=\ln 2\pi+\frac{\pi}{2}\cot\left(\frac{\pi s}{2}\right)-\psi(1-s)-\frac{\zeta'(1-s)}{\zeta(1-s)}$$

$$\textit{Evaluating limit at } s\longrightarrow 2:\textit{ let } s=2+\epsilon$$

$$\frac{\zeta'(2+\epsilon)}{\zeta(2+\epsilon)}=\ln 2\pi+\frac{\pi}{2}\cot\left(\pi+\frac{\pi\epsilon}{2}\right)-\psi(-1-\epsilon)-\frac{\zeta'(-1-\epsilon)}{\zeta(-1-\epsilon)}\Rightarrow\frac{\zeta'(2+\epsilon)}{\zeta(2+\epsilon)}$$

$$=\ln 2\pi+\frac{\pi}{2}\left(\frac{2}{\pi\epsilon}-\frac{\pi\epsilon}{6}+O(\epsilon^3)\right)-\psi(-1-\epsilon)-\frac{\zeta'(-1-\epsilon)}{\zeta(-1-\epsilon)}$$

$$\frac{\zeta'(2+\epsilon)}{\zeta(2+\epsilon)}=\ln 2\pi+\frac{1}{\epsilon}-\frac{\pi^2\epsilon}{12}+O(\epsilon^3)-\psi(-1-\epsilon)-\frac{\zeta'(-1-\epsilon)}{\zeta(-1-\epsilon)}$$

$$\psi(-1-\epsilon)=\psi(-\epsilon)+\frac{1}{1+\epsilon}\ ,\ \ \psi(-\epsilon)=\psi(1-\epsilon)+\frac{1}{\epsilon}$$

$$\frac{\zeta'(2+\epsilon)}{\zeta(2+\epsilon)}=\ln 2\pi+\frac{1}{\epsilon}-\frac{\pi^2\epsilon}{12}+O(\epsilon^3)-\left(\psi(1-\epsilon)+\frac{1}{\epsilon}+\frac{1}{1+\epsilon}\right)-\frac{\zeta'(-1-\epsilon)}{\zeta(-1-\epsilon)}$$

$$\frac{\zeta'(2+\epsilon)}{\zeta(2+\epsilon)}=\ln 2\pi+\frac{1}{\epsilon}-\frac{\pi^2\epsilon}{12}+O(\epsilon^3)-\left(-\gamma+\frac{1}{\epsilon}+1-\epsilon+O(\epsilon^2)\right)-\frac{\zeta'(-1-\epsilon)}{\zeta(-1-\epsilon)}$$

$$\Rightarrow\frac{\zeta'(2+\epsilon)}{\zeta(2+\epsilon)}=\ln 2\pi-\frac{\pi^2\epsilon}{12}+\gamma-1-\frac{\zeta'(-1-\epsilon)}{\zeta(-1-\epsilon)}$$

$$\frac{\zeta'(2)}{\zeta(2)}=\ln 2\pi+\gamma-1-\frac{\zeta'(-1)}{\zeta(-1)}$$

$$=\ln 2\pi+\gamma-1-\frac{\frac{1}{12}-\ln A}{\left(-\frac{1}{12}\right)}\overset{A:\ \textit{Glaisher–Kinkelin constant}}{\Longrightarrow}\ln 2\pi+\gamma-1$$

$$-(-1+12\ln A)$$

$$\zeta'(2)=\zeta(2)(\ln 2\pi+\gamma-12\ln A)=\frac{\pi^2}{6}(\gamma+\ln 2\pi-12\ln A)$$

$$\lambda'(2)=2^{-2}\ln 2\,\zeta(2)+(1-2^{-2})\zeta'(2)=\frac{\pi^2\ln 2}{24}+\frac{\pi^2(\gamma+\ln 2\pi-12\ln A)}{8}$$

$$\Omega=\frac{\pi^2(1-\gamma)}{4}+2\left(-\sum_{r=0}^{\infty}\frac{\ln\big((2r+1)\big)}{(2r+1)^2}\right)=\frac{\pi^2(1-\gamma)}{4}+2\lambda'(2)$$

$$\Omega=\frac{\pi^2(1-\gamma)}{4}+2\left(\frac{\pi^2\ln 2}{24}+\frac{\pi^2(\gamma+\ln 2\pi-12\ln A)}{8}\right)$$

$$\Omega=\frac{\pi^2(1-\gamma)}{4}+\frac{\pi^2\ln 2}{12}+\frac{\pi^2(\gamma+\ln 2\pi-12\ln A)}{4}$$

$$\boxed{\Omega=\pi^2\left(\frac{3+4\ln 2+3\ln\pi-36\ln A}{12}\right)}$$