

ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$I = \int_0^1 \int_0^1 \frac{\tan^{-1}\left(\frac{1}{xy}\right)}{1+x^2y^2} dx dy$$

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Change of variables: $xy = u, x = v : y = \frac{u}{v}$

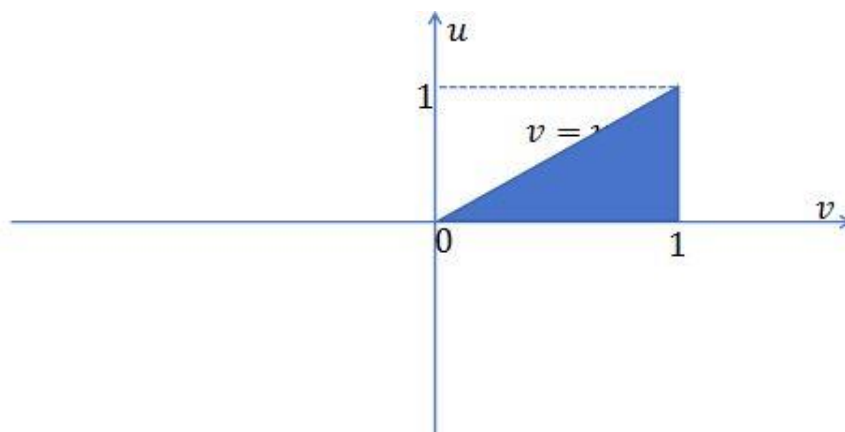
$$y \rightarrow 0, u \rightarrow 0; y \rightarrow 1, u \rightarrow v$$

$$x \rightarrow 0, v \rightarrow 0; x \rightarrow 1, v \rightarrow 1$$

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ 1 & 0 \end{vmatrix} = -x = -v$$

$$\therefore dudv = |-v|dxdy; dxdy = \frac{1}{v} dudv$$

$$I = \int_0^1 \frac{1}{v} \int_0^v \frac{\tan^{-1}\left(\frac{1}{u}\right)}{1+u^2} dudv$$



Change of order of integration: $0 \leq u \leq 1; u \leq v \leq 1$

$$I = \int_0^1 \int_u^1 \frac{1}{v} \frac{\tan^{-1}\left(\frac{1}{u}\right)}{1+u^2} dv du = \int_0^1 \frac{\ln\left(\frac{1}{u}\right) \tan^{-1}\left(\frac{1}{u}\right)}{1+u^2} du$$

$$t = \tan^{-1}\left(\frac{1}{u}\right), dt = -\frac{1}{1+u^2} du; t \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$$

$$I = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} t \ln(\tan(t)) dt$$

Fourier series for $\ln(\tan(t))$

$$= -2 \sum_{k=0}^{\infty} \frac{\cos(2(2k+1)t)}{2k+1} \quad \forall \quad 0 \leq t \leq \frac{\pi}{2}$$

$$\therefore I = -2 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} t \sum_{k=0}^{\infty} \frac{\cos(2(2k+1)t)}{2k+1} dt$$

Switch of order of sum and integral via L^2 Convergence

$$I = -2 \sum_{k=0}^{\infty} \frac{1}{2k+1} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} t \cos(2(2k+1)t) dt$$

By successive integration by parts

$$I = -2 \sum_{k=0}^{\infty} \frac{1}{2k+1} \left[\frac{t \sin(2(2k+1)t)}{2(2k+1)} + \frac{\cos(2(2k+1)t)}{4(2k+1)^2} \right] \Bigg|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$\text{NB: } \sin((2k+1)\pi) = 0, \cos\left((2k+1)\frac{\pi}{2}\right) = 0$$

$$\sin\left((2k+1)\frac{\pi}{2}\right) = (-1)^k, \cos((2k+1)\pi) = -1 : k \in \mathbb{Z}$$

$$\therefore I = - \sum_{k=0}^{\infty} \frac{1}{2k+1} \left[\frac{-\frac{\pi}{4}(-1)^k}{(2k+1)} - \frac{1}{2(2k+1)^2} \right]$$

$$I = \frac{\pi}{4} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} + \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^3}$$

$$\text{NB: } \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = \beta(2) = G \text{ (Catalan's constant)}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^3} = \zeta(3) \text{ (Apery's constant)}$$

$$\text{But: } \sum_{k=0}^{\infty} \frac{1}{(2k+1)^3} = \sum_{k=1}^{\infty} \frac{1}{k^3} - \sum_{k=1}^{\infty} \frac{1}{8k^3} = \frac{7}{8} \sum_{k=1}^{\infty} \frac{1}{k^3} = \frac{7}{8} \zeta(3)$$

$\therefore$

$$I = \frac{\pi}{4} G + \frac{7}{16} \zeta(3)$$