

ROMANIAN MATHEMATICAL MAGAZINE

Find :

$$\Delta = \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{1}{n^2} \left(\frac{1}{(4n+1)^2} + \frac{1}{(4n-1)^2} \right)$$

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Solution by Nawar Alasadi-Iraq

$$f(n) = \frac{1}{n^2(4n+1)^2} - \frac{1}{n^2(4n-1)^2}$$

$$f(n) = \frac{2}{n^2} + 32 \left(\frac{1}{4n+1} - \frac{1}{4n-1} \right) + 16 \left(\frac{1}{(4n+1)^2} + \frac{1}{(4n-1)^2} \right)$$

$$\sum_{n \in \mathbb{Z} \setminus \{0\}} f(n) = 2 \sum_{n=1}^{\infty} f(n), \quad \sum_{n=1}^{\infty} \frac{2}{n^2} = \frac{\pi^2}{3}$$

$$32 \sum_{n=1}^{\infty} \left(\frac{1}{4n+1} - \frac{1}{4n-1} \right) = 32 \left(1 - \frac{\pi}{4} \right) = 8\pi - 32$$

$$16 \sum_{n=1}^{\infty} \left(\frac{1}{(4n+1)^2} + \frac{1}{(4n-1)^2} \right) = 16 \left(\frac{\pi^2}{8} - 1 \right) = 2\pi^2 - 16$$

$$2 \sum_{n=1}^{\infty} f(n) = 2 \left(\frac{\pi^2}{3} + 8\pi - 32 + 2\pi^2 - 16 \right) = \frac{14\pi^2}{3} + 16\pi - 96$$