ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\int_0^{\pi}\left|\frac{\sin x+\cos x}{\sin\frac{x}{2}+\cos\frac{x}{2}}\right|dx$$

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$$\sin\frac{5\pi}{8}=\cos\frac{\pi}{8}=\frac{\sqrt{2+\sqrt{2}}}{2},\cos\frac{5\pi}{8}=-\sin\frac{\pi}{8}=-\frac{\sqrt{2-\sqrt{2}}}{2},\tan\frac{\pi}{8}=\sqrt{2}-1$$

$$\tan\frac{3\pi}{8}=\cot\frac{\pi}{8}=\sqrt{2}+1,\tan^2\frac{5\pi}{16}=\frac{1-\cos\frac{5\pi}{8}}{1+\cos\frac{5\pi}{8}}=\frac{1+\frac{\sqrt{2-\sqrt{2}}}{2}}{1-\frac{\sqrt{2-\sqrt{2}}}{2}}=\frac{2+\sqrt{2-\sqrt{2}}}{2-\sqrt{2-\sqrt{2}}}$$

$$\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)=\sqrt{2}\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)\text{ always postive in}(0,\pi)\text{ as }0\le\frac{x}{2}\le\frac{\pi}{2}$$

$$\text{and }\sin x+\cos x=\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)>0\text{ when }0\le x<\frac{3\pi}{4}$$

$$\text{and }<0\text{ when }\frac{3\pi}{4}<x<\pi$$

$$\text{then}\left|\frac{\sin x+\cos x}{\sin\frac{x}{2}+\cos\frac{x}{2}}\right|>0\text{ in }\left(0,\frac{3\pi}{4}\right)\text{ and }<0\text{ in }\left(\frac{3\pi}{4},\pi\right)$$

$$\int_0^{\pi}\left|\frac{\sin x+\cos x}{.\sin\frac{x}{2}+\cos\frac{x}{2}}\right|dx=\int_0^{\frac{3\pi}{4}}\frac{\sin x+\cos x}{\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)}dx-\int_{\frac{3\pi}{4}}^{\pi}\frac{\sin x+\cos x}{\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)}dx\ (1)$$

$$\int\frac{\sin x+\cos x}{\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)}dx=\int\frac{\sin\left(x+\frac{\pi}{4}\right)}{\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)}dx\overset{u=\frac{x}{2}+\frac{\pi}{4}\text{ then }dx=2du}{=}2\int\frac{\sin\left(2u-\frac{\pi}{4}\right)}{\sin u}du=$$

$$=\frac{2}{\sqrt{2}}\int\left(\frac{\sin 2u}{\sin u}-\frac{\cos 2u}{\sin u}\right)du=\sqrt{2}\int(2\cos u+2\sin u-\csc u)du=$$

$$=\sqrt{2}\left(2\sin u-2\cos u-\log\tan\left(\frac{u}{2}\right)\right)$$

$$\text{Using this result we get:}$$

$$\int_0^{\frac{3\pi}{4}}\frac{\sin x+\cos x}{\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)}dx=\sqrt{2}\left(2\sin u-2\cos u-\log\tan\left(\frac{u}{2}\right)\right)_{\frac{\pi}{4}}^{\frac{5\pi}{8}}=$$

$$= \sqrt{2}\left(\log(\sqrt{2}-1) + \sqrt{2-\sqrt{2}} + \sqrt{2+\sqrt{2}} - \log\tan\left(\frac{5\pi}{16}\right)\right)$$

$$\int_{\frac{3\pi}{4}}^{\pi} \frac{\sin x + \cos x}{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)} dx = \sqrt{2}\left(2sinu - 2cosu - \log\tan\left(\frac{u}{2}\right)\right)_{\frac{5\pi}{8}}^{\frac{3\pi}{4}} =$$

$$= \sqrt{2}\left(2\sqrt{2} - \log(\sqrt{2}+1) - \sqrt{2-\sqrt{2}} - \sqrt{2+\sqrt{2}} + \log\tan\left(\frac{5\pi}{16}\right)\right)$$

$$\textit{From (1) we get}$$

$$\int_0^{\pi} \left|\frac{\sin x + \cos x}{sin\frac{x}{2} + \cos\frac{x}{2}}\right| dx =$$

$$= \sqrt{2}\left(\log(\sqrt{2}-1) + \sqrt{2-\sqrt{2}} + \sqrt{2+\sqrt{2}} - \log\tan\left(\frac{5\pi}{16}\right)\right)$$
$$- \sqrt{2}\left(2\sqrt{2} - \log(\sqrt{2}+1) - \sqrt{2-\sqrt{2}} - \sqrt{2+\sqrt{2}} + \log\tan\left(\frac{5\pi}{16}\right)\right)$$
$$= \sqrt{2}\left(-2\sqrt{2} + 2\sqrt{2-\sqrt{2}} + 2\sqrt{2+\sqrt{2}} + \log\frac{1}{\tan^2\frac{5\pi}{16}}\right) =$$

$$= \sqrt{2}\left(-2\sqrt{2} + 2\sqrt{2-\sqrt{2}} + 2\sqrt{2+\sqrt{2}} + \log\frac{2-\sqrt{2-\sqrt{2}}}{2+\sqrt{2-\sqrt{2}}}\right)$$