

ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$I = \int_0^{+\infty} \frac{dx}{\sqrt{x^7 + 1}}$$

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$$I = \int_0^{+\infty} \frac{dx}{\sqrt{x^7 + 1}}$$

$$x^7 = t \Rightarrow x = t^{\frac{1}{7}} \Rightarrow dx = \frac{1}{7} t^{-\frac{6}{7}} dt$$

$$\begin{aligned} I &= \frac{1}{7} \int_0^{+\infty} \frac{t^{-\frac{6}{7}}}{\sqrt{t+1}} dt \\ &= \frac{1}{7} \int_0^{+\infty} \frac{t^{\frac{1}{7}-1}}{(1+t)^{\frac{1}{7}+\frac{5}{14}}} dt \end{aligned}$$

$$B(x, y) = \int_0^{+\infty} \frac{u^{x-1}}{(1+u)^{x+y}} du = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

$$\begin{aligned} I &= \frac{1}{7} B\left(\frac{1}{7}, \frac{5}{14}\right) \\ &= \frac{1}{7} \frac{\Gamma\left(\frac{1}{7}\right) \Gamma\left(\frac{5}{14}\right)}{\Gamma\left(\frac{1}{7} + \frac{5}{14}\right)} \\ &= \frac{\Gamma\left(\frac{1}{7}\right) \Gamma\left(\frac{5}{14}\right)}{7 \Gamma\left(\frac{1}{2}\right)} \\ &= \frac{\Gamma\left(\frac{1}{7}\right) \Gamma\left(\frac{5}{14}\right)}{7\sqrt{\pi}} \end{aligned}$$