ROMANIAN MATHEMATICAL MAGAZINE

*Find :*

$$\int_0^{\frac{\pi}{6}} \frac{\sin^2(x)}{(\sin(x)+2\cos(x))^2}dx$$

*Proposed by Nguyen Hung Cuong-Vietnam*

*Solution by Mirsadix Muzefferov-Azerbaijan*

$$\frac{\sin^2(x)}{(\sin(x)+2\cos(x))^2} = \frac{\sin^2(x)}{\sin^2(x)+4\sin(x)\cos(x)+4\cos^2(x)} =$$

$$= \frac{\tan^2(x)}{(\tan(x)+2)^2}, \quad \begin{cases} \tan(x) \to t \\ dx = \dfrac{dt}{1+t^2} \end{cases}$$

$$\Omega = \int_0^{\frac{\pi}{6}} \frac{\sin^2(x)}{(\sin(x)+2\cos(x))^2}dx = \int_0^{\frac{\sqrt{3}}{3}} \frac{t^2}{(t+2)^2} \cdot \frac{dt}{t^2+1}$$

$$\frac{t^2}{(t+2)^2} = \frac{A}{(t+2)^2} + \frac{B}{t+2} + \frac{Ct+D}{t^2+1}$$

$$t^2 = A(t^2+1) + B(t+2)(t^2+1) + (Ct+D)(t+2)^2$$

$$\begin{cases} B+C=0 \\ A+2B+4C+D=1 \\ B+4C+4D=0 \\ A+2B+4D=0 \end{cases} \to \begin{cases} A=\dfrac{4}{5}, \quad B=-\dfrac{4}{25} \\ C=\dfrac{4}{25}, \quad D=-\dfrac{3}{25} \end{cases}$$

$$\Omega = \Omega_1 + \Omega_2 + \Omega_3$$

$$\Omega_1 = \int_0^{\frac{\sqrt{3}}{3}} \frac{\frac{4}{5}}{(t+2)^2}dt = -\frac{4}{5}\cdot\frac{1}{t+2}\Big|_0^{\frac{\sqrt{3}}{3}} = \frac{2(2\sqrt{3}-1)}{55}$$

$$\Omega_2 = \int_0^{\frac{\sqrt{3}}{3}} \frac{-\frac{4}{25}}{t+2}dt = -\frac{4}{25}.\ln(t+2)\Big|_0^{\frac{\sqrt{3}}{3}} = -\frac{4}{25}\ln\left(\frac{6+\sqrt{3}}{6}\right)$$

$$\Omega_3 = \int_0^{\frac{\sqrt{3}}{3}} \frac{\frac{4}{25}t-\frac{3}{25}}{t^2+1}dt = \frac{1}{25}\int_0^{\frac{\sqrt{3}}{3}} \frac{4t-3}{t^2+1}dt = \frac{2}{25}\int_0^{\frac{\sqrt{3}}{3}} \frac{d(t^2+1)}{t^2+1} =$$

$$\frac{2}{25}.\ln(t^2+1)\Big|_0^{\frac{\sqrt{3}}{3}} - \frac{3}{25}.\arctan(t)\Big|_0^{\frac{\sqrt{3}}{3}} = \frac{2}{25}\ln\left(\frac{4}{3}\right) - \frac{\pi}{50}$$

$\boldsymbol{Then}$:

$$\Omega=\Omega_1+\Omega_2+\Omega_3=\frac{2\left(2\sqrt{3}-1\right)}{55}-\frac{4}{25}\ln\left(\frac{6+\sqrt{3}}{6}\right)+\frac{2}{25}\ln\left(\frac{4}{3}\right)-\frac{\pi}{50}=$$

$$\frac{4\sqrt{3}-2}{55}+\frac{4}{25}ln\left(\frac{4}{2\sqrt{3}+1}\right)-\frac{\pi}{50}=$$

$$\frac{4\sqrt{3}-2}{55}+\frac{8}{25}\ln(2)-\frac{4}{25}\ln\left(2\sqrt{3}+1\right)-\frac{\pi}{50}$$