ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega = \int_0^\infty \frac{dx}{x\sqrt{x}+1}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution 1 by Daniel Sitaru – Romania

$$\Omega = \int_0^\infty \frac{dx}{x\sqrt{x}+1} \overset{(x=y^2)}{=} \int_0^\infty \frac{2ydy}{y^3+1}$$

$$\frac{2y}{(y+1)(y^2-y+1)} = \frac{A}{y+1} + \frac{By+C}{y^2-y+1}, \quad 2y = A(y^2-y+1)+(y+1)(By+C)$$

$$2y = Ay^2 - Ay + A + By^2 + Cy + By + C, 2y = (A+B)y^2 + y(-A+B+C) + A + C$$

$$\begin{cases} A+B=0 \\ -A+B+C=2 \\ A+C=0 \end{cases} \Rightarrow B=-A \Rightarrow \begin{cases} -2A+C=2 \\ 2A+C=0 \end{cases} \Rightarrow 3C=2 \Rightarrow C=\frac{2}{3} \Rightarrow A=-\frac{2}{3} \Rightarrow B=\frac{2}{3}$$

$$\Omega = \int_0^\infty \left(\frac{-\frac{2}{3}}{y+1} + \frac{\frac{2}{3}(y+1)}{y^2-y+1}\right)dy = -\frac{2}{3}\ln(y+1)\Big|_0^\infty + \frac{1}{3}\int_0^\infty \frac{2y-1+3}{y^2-y+1}dy =$$

$$= -\frac{2}{3}\ln(y+1) + \frac{1}{3}\ln(y^2-y+1)\Big|_0^\infty + \int_0^\infty \frac{1}{\left(y-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{1}{3}\ln\left(\frac{y^2-y+1}{y^2+2y+1}\right)\Big|_0^\infty + \frac{1}{\frac{\sqrt{3}}{2}}\arctan\left(\frac{y-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\Bigg|_0^\infty =$$

$$= \frac{1}{3}(\ln 1 - \ln 1) + \frac{2}{\sqrt{3}}\left(\arctan\infty - \arctan\left(-\frac{1}{\sqrt{3}}\right)\right) =$$

$$= \frac{1}{3}(0-0) + \frac{2}{\sqrt{3}}\left(\frac{\pi}{2}+\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}}\cdot\frac{4\pi}{6} = \frac{8\pi}{6\sqrt{3}} = \frac{4\pi}{3\sqrt{3}}$$

Solution 2 by Daniel Sitaru – Romania

$$\int_0^\infty \frac{dx}{x^p+1} = \frac{\pi}{p\sin\left(\frac{\pi}{p}\right)}$$

$$\Omega = \int_0^\infty \frac{dx}{x^{\frac{2}{3}} + 1} = \frac{\pi}{\frac{3}{2}\sin\left(\frac{\pi}{\frac{3}{2}}\right)} = \frac{2\pi}{3\sin\frac{2\pi}{3}} = \frac{2\pi}{3\sin\left(\pi - \frac{\pi}{3}\right)} = \frac{2\pi}{3\cdot\sin\frac{\pi}{3}} = \frac{2\pi}{\frac{3\sqrt{3}}{2}} = \frac{4\pi}{3\sqrt{3}}$$