ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\sec(x)+\cos^3(x)}\,dx$$

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$$I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\sec(x)+\cos^3(x)}\,dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos(x)}{1+\cos^4(x)}\,dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos(x)}{1+\left(1-\sin^2(x)\right)^2}\,dx$$

$$\textit{substitution } \sin(x)=t \quad \frac{dt}{dx}=\cos(x),\ \ t\to\frac{\sqrt{3}}{2};\frac{1}{2}$$

$$I=\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{t^4-2t^2+2}\,dt\to\ x^4-2t^2+2=(x^2+a)(x^2+b)\to\begin{cases}a+b=-2\\ ab=2\end{cases}$$

$$a=-(1-i)\quad b=-(1+i)$$

$$\frac{1}{(x^2+a)(x^2+b)}=\frac{A}{x^2+a}+\frac{B}{x^2+b}\to\begin{cases}A+B=0\\ Ab+Ba=1\end{cases}\to\begin{cases}A=\dfrac{1}{b-a}\\ B=\dfrac{1}{a-b}\end{cases}$$

$$I_{a;b}=\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{(x^2+a)(x^2+b)}\,dx=\frac{1}{b-a}\left(\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{x^2+a}\,dx-\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{x^2+b}\,dx\right)$$
$$=\frac{I(b)-I(a)}{a-b}$$

$$I(a)=\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{x^2+a}\,dx=\frac{1}{a}\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{\left(\frac{x}{\sqrt{a}}\right)^2+1}\,dx=\left[\frac{1}{\sqrt{a}}\arctan\left(\frac{x}{\sqrt{a}}\right)\right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}$$
$$=\frac{\tan^{-1}\left(\frac{1}{2}\sqrt{\frac{3}{a}}\right)-\tan^{-1}\left(\frac{1}{2\sqrt{a}}\right)}{\sqrt{a}}$$

$$I(b)=\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}\frac{1}{x^2+b}dx=\frac{\tan^{-1}\left(\frac{1}{2}\sqrt{\frac{3}{b}}\right)-\tan^{-1}\left(\frac{1}{2\sqrt{b}}\right)}{\sqrt{b}}\quad a=i-1;\quad b=-i-1$$

$$I=\frac{\tan^{-1}\left(\frac{1}{2}\sqrt{\frac{3}{-i-1}}\right)-\tan^{-1}\left(\frac{1}{2\sqrt{-i-1}}\right)}{2i\sqrt{-i-1}}-\frac{\tan^{-1}\left(\frac{1}{2}\sqrt{\frac{3}{i-1}}\right)-\tan^{-1}\left(\frac{1}{2\sqrt{i-1}}\right)}{2i\sqrt{i-1}}$$