ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\cot^2(x)+\tan^2(x)}dx$$

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$$I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\cot^2(x)+\tan^2(x)}dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin^2(x)\cos^2(x)}{\sin^4(x)+\cos^4(x)}dx=$$

$$=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin^2(x)\cos^2(x)}{1-2\sin^2(x)\cos^2(x)}dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\frac{\sin^2(2x)}{4}.}{1-\frac{\sin^2(2x)}{2}.}dx=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin^2(2x)}{2-\sin^2(2x)}dx$$

$$2x=t \quad dt=2dx \quad t\to\frac{2\pi}{3};\frac{\pi}{3}$$

$$I=\frac{1}{4}\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\sin^2(t)}{2-\sin^2(t)}dx=\frac{1}{4}\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1-\cos(2t)}{3+\cos(2t)}dt \quad 2t\to t$$

$$I=\frac{1}{8}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{4-(3+\cos(t))}{3+\cos(t)}dt=\frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{3+\cos(t)}dt-\frac{1}{8}\cdot\frac{2\pi}{3}=\frac{1}{2}K-\frac{\pi}{12}$$

$$K=\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{3+\cos(t)}dt \quad \to \quad \cos(t)=\frac{1-\tan^2\left(\frac{t}{2}\right)}{1+\tan^2\left(\frac{t}{2}\right)}$$

The Weierstrass Substitution

$$\tan\left(\frac{t}{2}\right)=u \quad \frac{du}{dt}=\frac{1}{2}\sec^2\left(\frac{t}{2}\right)=\frac{1}{1+\cos(t)}=\frac{1+\tan^2\left(\frac{t}{2}\right)}{2}=\frac{1+u^2}{2}$$

$$K(a,b)=\int\frac{1}{a+b\cos(t)}dt \overset{\tan\left(\frac{t}{2}\right)=u}{\triangleq} \int\frac{1}{a+b\frac{1-u^2}{1+u^2}}\frac{2du}{u^2+1}$$

$$=2\int\frac{1}{a+au^2+b-bu^2}du=$$

$$=2\int\frac{1}{u^2(a-b)+a+b}du=\frac{2}{a-b}\int\frac{1}{u^2+\left(\frac{a+b}{a-b}\right)^2}du$$

$$=2\left[\frac{\arctan\left(\frac{u\sqrt{a-b}}{\sqrt{a+b}}\right)}{\sqrt{a^2-b^2}}\right]_{u=\tan\left(\frac{t}{2}\right)}+C$$

$$=\frac{2}{\sqrt{a^2-b^2}}\arctan\left(\frac{\tan\left(\frac{t}{2}\right)\sqrt{a-b}}{\sqrt{a+b}}\right)+C\to\ a=3\ b=1$$

$$K(3,1)=\frac{1}{\sqrt{2}}\arctan\left(\frac{\sqrt{2}\,tan\left(\frac{t}{2}\right)}{2}\right)+C\quad t\to\tan\left(\frac{2\pi}{3}\right)=-\sqrt{3},\tan\left(\frac{\pi}{3}\right)=\sqrt{3}$$

$$K=\frac{2}{\sqrt{3}}\left[\arctan\left(\frac{\sqrt{2}\,tan\left(\frac{t}{2}\right)}{2}\right)\right]_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}=\frac{1}{\sqrt{2}}\left(-\tan^{-1}\sqrt{\frac{3}{2}}-\tan^{-1}\sqrt{\frac{3}{2}}\right)$$

Note that t = π is a point of discontinuity for $tan\left(\frac{t}{2}\right)$ within the given range. Thus, the antiderivative must be adjusted for continuity, yielding:

$$K=\frac{1}{\sqrt{2}}\left(\pi-2\arctan\sqrt{\frac{3}{2}}\right)$$

$$Therefore\ \ I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{1}{\cot^2(x)+\tan^2(x)}dx=\frac{1}{2}K-\frac{\pi}{12}=\frac{1}{2\sqrt{2}}\left(\pi-2\arctan\sqrt{\frac{3}{2}}\right)-\frac{\pi}{12}$$