ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin(x)}{\cot(x)\,(1+\cos(x))}dx$$

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$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin(x)}{\cot(x)\,(1+\cos(x))}dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin^2(x)}{\cos(x)\,(1+\cos(x))}dx =$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(1-\cos(x))(1+\cos(x))}{\cos(x)\,(1+\cos(x))}dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1-\cos(x)}{\cos(x)}dx =$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\cos(x)}dx - \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\cos(x)}dx - \frac{\pi}{6} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1+\tan^2\left(\frac{x}{2}\right)}{1-\tan^2\left(\frac{x}{2}\right)}dx - \frac{\pi}{6}$$

$$\left\{\tan\left(\frac{x}{2}\right) = t\,, \frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2} = \frac{1+\tan^2\left(\frac{x}{2}\right)}{2}, t \in \left[\frac{1}{\sqrt{3}}; 2-\sqrt{3}\right]\right\}$$

$$I = 2\int_{2-\sqrt{3}}^{\frac{1}{\sqrt{3}}} \frac{dt}{1-t^2} - \frac{\pi}{6} = \left[\ln\left(\frac{1+t}{1-t}\right)\right]_{2-\sqrt{3}}^{\frac{1}{\sqrt{3}}} - \frac{\pi}{6} = \ln\left(\frac{2+\sqrt{3}}{\sqrt{3}}\right) - \frac{\pi}{6}$$

$$\textit{Therefore}:$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin(x)}{\cot(x)(1+\cos(x))}dx = \ln\left(\frac{2\sqrt{3}+3}{3}\right) - \frac{\pi}{6}$$