ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{ctg^3(x)}{1+sin^2(x)}dx$$

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Solution by Sahel Koley-India

$$I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{ctg^3(x)}{1+sin^2(x)}dx=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{cos^3(x)}{\sin(x)\left(1+sin^2(x)\right)}dx=$$

$$=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos(x)\left(1-sin^2(x)\right)}{\sin(x)\left(1+sin^2(x)\right)}dx$$

$$Let\ \sin(x)=t,\qquad dt=\cos(x)\,dx$$

$$\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1-t^2}{t(1+t^2)}dt=\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1+t^2-2t^2}{t(1+t^2)}dt=\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}\left(\frac{1}{t}-\frac{2t}{1+t^2}\right)dt=\left|\ln(t)-\ln(1+t^2)\right|_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}=$$

$$=\ln\left(\frac{\frac{\sqrt{3}}{2}}{1+\left(\frac{\sqrt{3}}{2}\right)^2}\right)-\ln\left(\frac{0.5}{1+0.25}\right)=\ln\left(\frac{5\sqrt{3}}{7}\right)$$