Find:

$$\int_{-\infty}^{+\infty} \frac{e^x}{e^{4x}+1}\,dx$$

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Solution by Dau Trung-Vietnam

$$\int_{-\infty}^{+\infty} \frac{e^x}{e^{4x}+1}\,dx = \int_{-\infty}^{+\infty} \frac{1}{e^{4x}+1}\,d(e^x) \overset{t=e^x}{=\!=}$$

$$\int_0^{+\infty} \frac{1}{t^4+1}\,d(t) \quad t^4 = \frac{u}{1-u}$$

$$= \frac{1}{4}\int_0^1 u^{-\frac{3}{4}}(1-u)^{-\frac{1}{4}}\,du = \frac{1}{4}\beta\left(\frac{1}{4},\frac{3}{4}\right) = \frac{1}{4}\frac{\Gamma\left(\frac{1}{4}\right)\cdot\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}+\frac{3}{4}\right)} = \frac{1}{4}\cdot\Gamma\left(\frac{1}{4}\right)\cdot\Gamma\left(1-\frac{1}{4}\right) = \frac{\pi\sqrt{2}}{4}$$