ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\int_0^{\frac{\pi}{6}} \frac{\sin(2x)}{1+\sin(3x)}dx$$

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$$\Omega = \int_0^{\frac{\pi}{6}} \frac{\sin(2x)}{1+\sin(3x)}dx = \int_0^{\frac{\pi}{6}} \frac{2\sin(x)\cos(x)}{1+3\sin(x)-4\sin^3(x)}dx =$$

$$\int_0^{\frac{\pi}{6}} \frac{2\sin(x)}{(1-\sin(x))(1+2\sin(x))^2}d\sin(x) \overset{\sin(x)\to t}{\triangleq} \int_0^{\frac{1}{2}} \frac{2t}{(1-t)(1+2t)^2}dt$$

$$\frac{2t}{(1-t)(1+2t)^2} = \frac{A}{1-t} + \frac{B}{1+2t} + \frac{C}{(1+2t)^2}$$

$$2t = A(1+2t)^2 + B(1-t)(1+2t) + C(1-t)$$

$$A = \frac{2}{9}, \quad B = \frac{4}{9}, \quad C = -\frac{2}{3}$$

$$\int_0^{\frac{1}{2}} \frac{2t}{(1-t)(1+2t)^2}dt = \frac{2}{9}\int_0^{\frac{1}{2}} \frac{dt}{1-t} + \frac{4}{9}\int_0^{\frac{1}{2}} \frac{dt}{1+2t} - \frac{2}{3}\int_0^{\frac{1}{2}} \frac{dt}{(1+2t)^2} = \Omega_1 + \Omega_2 + \Omega_3$$

$$\Omega_1 = \frac{2}{9}\int_0^{\frac{1}{2}} \frac{dt}{1-t} = -\frac{2}{9}ln|1-t|\Big|_0^{\frac{1}{2}} = -\frac{2}{9}\ln\left(\frac{1}{2}\right) = \frac{2}{9}\ln(2)$$

$$\Omega_2 = \frac{4}{9}\int_0^{\frac{1}{2}} \frac{dt}{1+2t} = \frac{2}{9}ln|1+2t|\Big|_0^{\frac{1}{2}} = \frac{2}{9}\ln(2)$$

$$\Omega_3 = -\frac{2}{3}\int_0^{\frac{1}{2}} \frac{dt}{(1+2t)^2} = -\frac{2}{3}\left|-\frac{1}{2(1+2t)}\right|\Big|_0^{\frac{1}{2}} = -\frac{1}{6}$$

$$So: \ \Omega = \Omega_1 + \Omega_2 + \Omega_3 = \frac{2}{9}\ln(2) + \frac{2}{9}\ln(2) - \frac{1}{6} = \frac{4}{9}\ln(2) - \frac{1}{6}$$