ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\Omega = \int_0^{\frac{\pi}{6}} \frac{\cos^3(x)}{(1+\cos(x))^2} dx$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution 1 by Amin Hajiyev-Azerbaijan

$$I = \int_0^{\frac{\pi}{6}} \frac{\cos^3(x)}{(1+\cos(x))^2} dx = \int_0^{\frac{\pi}{6}} \frac{\cos^3(x)+1}{(1+\cos(x))^2} dx - \int_0^{\frac{\pi}{6}} \frac{dx}{(1+\cos(x))^2} = I_1 - I_2$$

$$I_1 = \int_0^{\frac{\pi}{6}} \frac{(\cos(x)+1)(\cos^2(x)-\cos(x)+1)}{(1+\cos(x))^2} dx = \int_0^{\frac{\pi}{6}} \frac{(\cos(x)+1)(\cos(x)-2)+3}{1+\cos(x)} dx =$$

$$= \int_0^{\frac{\pi}{6}} (\cos(x)-2)dx + 3\int_0^{\frac{\pi}{6}} \frac{1}{1+\cos(x)} dx = [\sin(x)-2x]_0^{\frac{\pi}{6}} + \frac{3}{2}\int_0^{\frac{\pi}{6}} \sec^2\left(\frac{x}{2}\right) dx =$$

$$= \sin\left(\frac{\pi}{6}\right) - \frac{\pi}{3} + \frac{3}{2}\left[2\tan\left(\frac{x}{2}\right)\right]_0^{\frac{\pi}{6}} = \frac{1}{2} - \frac{\pi}{3} + 3\underbrace{\tan\left(\frac{\pi}{12}\right)}_{=2-\sqrt{3}} = \frac{13}{2} - \frac{\pi}{3} - 3\sqrt{3}$$

$$I_2 = \int_0^{\frac{\pi}{6}} \frac{dx}{(1+\cos(x))^2} = \frac{1}{4}\int_0^{\frac{\pi}{6}} \sec^4\left(\frac{x}{2}\right) dx = \frac{1}{4}\int_0^{\frac{\pi}{6}} \sec^2\left(\frac{x}{2}\right)\left(1+\tan^2\left(\frac{x}{2}\right)\right) dx =$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{6}} \sec^2\left(\frac{x}{2}\right) dx + \underbrace{\frac{1}{4}\int_0^{\frac{\pi}{6}} \sec^2\left(\frac{x}{2}\right)\tan^2\left(\frac{x}{2}\right) dx}_{t=\tan\left(\frac{x}{2}\right)\to\frac{dt}{dx}=\frac{1}{2}\sec^2\left(\frac{x}{2}\right)} = \frac{1}{2}\left[\tan\left(\frac{x}{2}\right)\right]_0^{\frac{\pi}{6}} + \frac{1}{2}\int_0^{2-\sqrt{3}} t^2 dt =$$

$$= \frac{1}{2}\tan\left(\frac{\pi}{12}\right) + \frac{1}{6}[t^3]_0^{2-\sqrt{3}} = \frac{2-\sqrt{3}}{2} + \frac{\left(2-\sqrt{3}\right)^3}{6} = \frac{16}{3} - 3\sqrt{3}$$

$$\int_0^{\frac{\pi}{6}} \frac{\cos^3(x)}{(1+\cos(x))^2} dx = I_1 - I_2 = \frac{13}{2} - \frac{\pi}{3} - 3\sqrt{3} - \frac{16}{3} + 3\sqrt{3} = \frac{7}{6} - \frac{\pi}{3}$$

Solution 2 by Mais Hasanov-Azerbaijan

$$I = \int_0^{\frac{\pi}{6}} \frac{\cos^3(x)}{(1+\cos(x))^2} dx = \int_0^{\frac{\pi}{6}} \frac{(2\cos^2(x)-1)^3}{(1+\cos(x))^2} dx =$$

$$=\int_0^{\frac{\pi}{6}}\frac{8\cos^6\left(\frac{x}{2}\right)-12\cos^4\left(\frac{x}{2}\right)+6\cos^2\left(\frac{x}{2}\right)-1}{4\cos^4\left(\frac{x}{2}\right)}dx=$$

$$=2\int_0^{\frac{\pi}{6}}\cos^2\left(\frac{x}{2}\right)dx-3\int_0^{\frac{\pi}{6}}dx+\frac{3}{2}\int_0^{\frac{\pi}{6}}\sec^2\left(\frac{x}{2}\right)dx-\frac{1}{4}\int_0^{\frac{\pi}{6}}\sec^4\left(\frac{x}{2}\right)dx=$$

$$=2\int_0^{\frac{\pi}{6}}\frac{1+\cos(x)}{2}dx-\frac{\pi}{2}+\frac{3}{2}\left[2\tan\left(\frac{x}{2}\right)\right]_0^{\frac{\pi}{6}}-\frac{1}{4}\int_0^{\frac{\pi}{6}}\sec^2\left(\frac{x}{2}\right)\left(1+\tan^2\left(\frac{x}{2}\right)\right)dx=$$

$$=\frac{\pi}{6}+[\sin(x)]_0^{\frac{\pi}{6}}-\frac{\pi}{2}+3\tan\left(\frac{\pi}{12}\right)-\frac{1}{4}\int_0^{\frac{\pi}{6}}\sec^2\left(\frac{x}{2}\right)dx-\frac{1}{4}\int_0^{\frac{\pi}{6}}\sec^2\left(\frac{x}{2}\right)\tan^2\left(\frac{x}{2}\right)dx=$$

$$=-\frac{\pi}{3}+\frac{1}{2}+3(2-\sqrt{3})-\frac{1}{2}\tan\left(\frac{\pi}{12}\right)-\frac{1}{4}K=-\frac{\pi}{3}+\frac{1}{2}+6-3\sqrt{3}-1+\frac{\sqrt{3}}{2}-\frac{1}{4}K=$$

$$=-\frac{\pi}{3}+\frac{11}{2}-\frac{5\sqrt{3}}{2}-\frac{1}{4}K$$

$$K=\int_0^{\frac{\pi}{6}}\sec^2\left(\frac{x}{2}\right)\tan^2\left(\frac{x}{2}\right)dx\to\tan\left(\frac{x}{2}\right)=u\ \frac{du}{dx}=\frac{1}{2}\sec^2\left(\frac{x}{2}\right)$$

$$K=2\int_0^{2-\sqrt{3}}u^2\,du=\frac{2}{3}[u^3]_0^{2-\sqrt{3}}=\frac{2\left(2-\sqrt{3}\right)^3}{3}=\frac{52}{3}-10\sqrt{3}$$

$$I=-\frac{\pi}{3}+\frac{11}{2}-\frac{5\sqrt{3}}{2}-\frac{13}{3}+\frac{5\sqrt{3}}{2}=\frac{7}{6}-\frac{\pi}{3}$$

$$\int_0^{\frac{\pi}{6}}\frac{\cos^3(x)}{(1+\cos(x))^2}dx=\frac{7}{6}-\frac{\pi}{3}$$