

ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\int_0^{\frac{\pi}{2}} \frac{(\sin x)^2 (\sin x - \cos x)^{2026}}{(\sin x + \cos x)^{2028}} dx$$

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$$\int_0^{\frac{\pi}{2}} \frac{(\sin x)^2 (\sin x - \cos x)^{2026}}{(\sin x + \cos x)^{2028}} dx \stackrel{x \rightarrow \frac{\pi}{2} - x}{\Rightarrow} \int_0^{\frac{\pi}{2}} \frac{(\cos x)^2 (\sin x - \cos x)^{2026}}{(\sin x + \cos x)^{2028}} dx$$

$$2I = \int_0^{\frac{\pi}{2}} \left(\frac{\sin x - \cos x}{\sin x + \cos x} \right)^{2026} \cdot \frac{1}{(\sin x + \cos x)^2} dx =$$

$$\stackrel{\frac{\sin x - \cos x}{\sin x + \cos x} \rightarrow y}{\Rightarrow} \int_{-1}^1 \frac{y^{2026}}{2} dy = \frac{1}{4054}$$

Observation:

$$\begin{aligned} dy &= \left(\frac{\sin x - \cos x}{\sin x + \cos x} \right)' dx \\ &= \frac{(\cos x + \sin x)(\sin x + \cos x) - (\sin x - \cos x)(\cos x - \sin x)}{(\sin x + \cos x)^2} dx = \\ &= \frac{(\sin x + \cos x)^2 + (\sin x - \cos x)^2}{(\sin x + \cos x)^2} dx = \frac{2}{(\sin x + \cos x)^2} dx \end{aligned}$$