ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\int_0^{\frac{\pi}{2}} \cos^2(x)\left(\sin(x)\sin^2\left(\frac{\pi}{2}\cos(x)\right)+\cos(x)\sin^2\left(\frac{\pi}{2}\sin(x)\right)\right)dx$$

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$$I=\int_0^{\frac{\pi}{2}} \cos^2(x)\left(\sin(x)\sin^2\left(\frac{\pi}{2}\cos(x)\right)+\cos(x)\sin^2\left(\frac{\pi}{2}\sin(x)\right)\right)dx=$$

$$=\int_0^{\frac{\pi}{2}} \cos^2(x)\left(\sin(x)\sin^2\left(\frac{\pi}{2}\cos(x)\right)\right)dx+\int_0^{\frac{\pi}{2}} \cos^2(x)\left(\cos(x)\sin^2\left(\frac{\pi}{2}\sin(x)\right)\right)dx$$

$$I=I_1+I_2.\textit{Using King Rule } \int_b^a f(x)dx=\int_b^a f(a+b-x)\,dx$$

$$\textit{Substitution } x=\frac{\pi}{2}-t \quad dx=-dt$$

$$I_1=\int_0^{\frac{\pi}{2}} \sin^2(t)\cos(t)\sin^2\left(\frac{\pi}{2}\sin(t)\right)dt,\qquad I_2=\int_0^{\frac{\pi}{2}} \sin^3(t)\sin^2\left(\frac{\pi}{2}\cos(t)\right)dt$$

$$2I=\int_0^{\frac{\pi}{2}} \sin^2(x)\left(\cos(x)\sin^2\left(\frac{\pi}{2}\sin(x)\right)+\sin(x)\sin^2\left(\frac{\pi}{2}\cos(x)\right)\right)dx$$
$$+\int_0^{\frac{\pi}{2}} \cos^2(x)\left(\sin(x)\sin^2\left(\frac{\pi}{2}\cos(x)\right)+\cos(x)\sin^2\left(\frac{\pi}{2}\sin(x)\right)\right)dx=$$

$$=\int_0^{\frac{\pi}{2}} \sin(x)\sin^2\left(\frac{\pi}{2}\cos(x)\right)dx+\int_0^{\frac{\pi}{2}} \cos(x)\sin^2\left(\frac{\pi}{2}\sin(x)\right)dx\ \left\{x\to\frac{\pi}{2}-x\right\}$$

$$I=\int_0^{\frac{\pi}{2}} \sin(x)\sin^2\left(\frac{\pi}{2}\cos(x)\right)dx\to\left\{\frac{\pi}{2}\cos(x)=t\ \ \frac{dt}{dx}=-\frac{\pi}{2}\sin(x)\right\}$$

$$I=\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin^2(t)\,dt=\frac{1}{\pi}\int_0^{\frac{\pi}{2}} dt-\frac{1}{\pi}\int_0^{\frac{\pi}{2}} \cos(2t)\,dt=\frac{1}{2}-\frac{1}{\pi}\left[\frac{1}{2}\sin(2t)\right]_0^{\frac{\pi}{2}}=\frac{1}{2}$$

$$\textit{Therefore } \int_0^{\frac{\pi}{2}} \cos^2(x)\left(\sin(x)\sin^2\left(\frac{\pi}{2}\cos(x)\right)+\cos(x)\sin^2\left(\frac{\pi}{2}\sin(x)\right)\right)dx=\frac{1}{2}$$