ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\int_0^{\infty} \frac{\cos(ln(x))}{1+x+x^2}dx$$

Proposed by Mais Hasanov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$Euler\ formula\ e^{ik} = \cos(k) + i\sin(k)\ \ \cos(k) = e^{ik} - i\sin(k)$$

$$k = \ln(x) \to \cos(ln(x)) = e^{i\ln x} - i\sin(ln(x)) = x^i - i\sin(\ln(x))$$

$$I = \int_0^{\infty} \frac{\cos(ln(x))}{1+x+x^2}dx = \int_0^{\infty} \frac{x^i}{1+x+x^2}dx - i\int_0^{\infty} \frac{\sin(ln(x))}{1+x+x^2}dx = I_1 - iI_2$$

$$I_2 = \int_0^{\infty} \frac{\sin(ln(x))}{1+x+x^2}dx \overset{x\to\frac{1}{x}}{\triangleq} -\int_0^{\infty} \frac{\sin(ln(x))}{1+x+x^2}dx = -I_2 \to 2I_2 = 0\,, \qquad I_2 = 0$$

$$I = \int_0^{\infty} \frac{x^i}{1+x+x^2}dx = \int_0^{1} \frac{x^i}{1+x+x^2}dx + \underbrace{\int_1^{\infty} \frac{x^i}{1+x+x^2}dx}_{x\to\frac{1}{x}} =$$

$$= \int_0^{1} \frac{x^i}{1+x+x^2}dx + \int_0^{1} \frac{x^{-i}}{1+x+x^2}dx$$

$$\to I(n) = \int_0^{1} \frac{x^n}{1+x+x^2}dx\,, \qquad \begin{cases} n_1 = i \\ n_2 = -i \end{cases} \to I = I(n_1) + I(n_2)$$

$$I(n) = \int_0^{1} \frac{x^n}{1+x+x^2}dx = \int_0^{1} \frac{x^n(1-x)}{1-x^3}dx, \qquad substitution\ x^3 = t,$$

$$\frac{dt}{dx} = 3x^2 = 3t^{\frac{2}{3}}$$

$$I(n) = \frac{1}{3}\int_0^{1} \frac{t^{\frac{n}{3}}\left(1 - t^{\frac{1}{3}}\right)}{(1-t)t^{\frac{2}{3}}}dt = \frac{1}{3}\left(\int_0^{1} \frac{t^{\frac{n-2}{3}} - t^{\frac{n-1}{3}}}{1-t}dt\right) = \frac{1}{3}\int_0^{1}\left(t^{\frac{n-2}{3}} - t^{\frac{n-1}{3}}\right)(1-t)^{-1}\,dt$$

$$Note: Beta\ function\ \int_0^{1} t^{a-1}(1-t)^{b-1}dt = \beta(a;b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

$$I(n)=\lim_{a\to 0^+}\frac{1}{3}\int_0^1\left(t^{\frac{n+1}{3}-1}-t^{\frac{n+2}{3}-1}\right)(1-t)^{a-1}\,dt=$$

$$=\lim_{a\to 0^+}\frac{1}{3}\left(\beta\left(\frac{n+1}{3};a\right)-\beta\left(\frac{n+2}{3};a\right)\right)=$$

$$=\lim_{a\to 0^+}\frac{1}{3}\left(\frac{\Gamma\left(\frac{n+1}{3}\right)\Gamma(a)}{\Gamma\left(\frac{n+1}{3}+a\right)}-\frac{\Gamma\left(\frac{n+2}{3}\right)\Gamma(a)}{\Gamma\left(\frac{n+2}{3}+a\right)}\right)$$

The Laurent series expansion of the Gamma function $\Gamma(a)$ about the simple pole at a=0 is given by: $\Gamma(a)=\frac{1}{a}-\gamma+\frac{1}{2}\left(\gamma^2+\frac{\pi^2}{6}\right)a+O(a^2)\sim\frac{1}{a}+O(1)$

The Taylor series expansion of the ratio $\frac{\Gamma(b)}{\Gamma(a+b)}$ around a = 0 is obtained by using the definition of the Digamma function $\psi^{(0)}(z)$: $\frac{\Gamma(b)}{\Gamma(a+b)}=1-\psi^{(0)}(b)a+O(a^2)$

$$I(n)=\frac{1}{3}\lim_{a\to 0^+}\Gamma(a)\left(\frac{\Gamma\left(\frac{n+1}{3}\right)}{\Gamma\left(\frac{n+1}{3}+a\right)}-\frac{\Gamma\left(\frac{n+2}{3}\right)}{\Gamma\left(\frac{n+2}{3}+a\right)}\right)=$$

$$=\frac{1}{3}\lim_{a\to 0^+}\left(\frac{1}{a}+O(1)\right)\left(1-\psi^{(0)}\left(\frac{n+1}{3}\right)a-1+\psi^{(0)}\left(\frac{n+2}{3}\right)a\right)=\frac{\psi^{(0)}\left(\frac{n+2}{3}\right)-\psi^{(0)}\left(\frac{n+1}{3}\right)}{3}$$

$$I=I(n_1)+I(n_2)=I(i)+I(-i)=$$

$$=\frac{\psi^{(0)}\left(\frac{i+2}{3}\right)-\psi^{(0)}\left(\frac{i+1}{3}\right)}{3}+\frac{\psi^{(0)}\left(\frac{-i+2}{3}\right)-\psi^{(0)}\left(\frac{-i+1}{3}\right)}{3}=$$

$$=\frac{1}{3}\left(\psi^{(0)}\left(1-\frac{1-i}{3}\right)-\psi^{(0)}\left(\frac{1-i}{3}\right)+\psi^{(0)}\left(1-\frac{1+i}{3}\right)-\psi^{(0)}\left(\frac{1+i}{3}\right)\right)$$

Digamma reflection formula: $\psi^{(0)}(1-z)-\psi^{(0)}(z)=\pi\cot(\pi z)$

$$I=\frac{1}{3}\left(\pi\cot\left(\pi\left(\frac{1-i}{3}\right)\right)+\pi\cot\left(\pi\left(\frac{i+1}{3}\right)\right)\right)=\frac{\pi}{3}\left(\cot\left(\frac{\pi}{3}-\frac{\pi i}{3}\right)+\cot\left(\frac{\pi}{3}+\frac{\pi i}{3}\right)\right)$$

$$\cot(a-b)+\cot(a+b)=\frac{\sin(2a)}{\sin^2(a)-\sin^2(b)}$$

$$I=\frac{\pi}{3}\frac{\sin\left(\frac{2\pi}{3}\right)}{\sin^2\left(\frac{\pi}{3}\right)-\sin^2\left(\frac{\pi i}{3}\right)}=\frac{\pi}{3}\frac{\frac{\sqrt{3}}{2}}{\frac{3}{4}-\left(i\sinh\left(\frac{\pi}{3}\right)\right)^2}=\frac{2\pi\sqrt{3}}{9+12\sinh^2\left(\frac{\pi}{3}\right)}=$$

$$=\frac{2\pi}{\sqrt{3}\left(3+4sinh^2\left(\frac{\pi}{3}\right)\right)}\to\left\{2\sinh^2\left(\frac{\pi}{3}\right)=\cosh\left(\frac{2\pi}{3}\right)-1\right\}$$

$$Therefore\ \int_0^\infty\frac{\cos(ln(x))}{1+x+x^2}dx=\frac{2\pi}{\sqrt{3}\left(1+2\cosh\left(\frac{2\pi}{3}\right)\right)}$$