

ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\int_0^{\infty} \frac{\ln(x) \ln(x^2 + 1)}{x^2} dx$$

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Using the following Frullani's integral

$$\begin{aligned} I &= \int_0^{\infty} \frac{\ln(x) \ln(x^2 + 1)}{x^2} dx = \\ &= -\frac{\ln(x) \ln(x^2 + 1)}{x} \Big|_0^{\infty} + \int_0^{\infty} \frac{1}{x} \left(\frac{\ln(x^2 + 1)}{x} + \frac{2x \ln(x)}{x^2 + 1} \right) dx = \\ &= 0 + \underbrace{\int_0^{\infty} \frac{\ln(x^2 + 1)}{x^2} dx}_A + \underbrace{\int_0^{\infty} \frac{2 \ln(x)}{x^2 + 1} dx}_B = A + B \\ A &= \int_0^{\infty} \frac{\ln(x^2 + 1)}{x^2} dx = -\frac{\ln(x^2 + 1)}{x} \Big|_0^{\infty} + \int_0^{\infty} \frac{2}{x^2 + 1} dx = 0 + 2 \arctan(x) \Big|_0^{\infty} = \pi \\ B &= 2 \int_0^{\infty} \frac{\ln(x)}{x^2 + 1} dx = \int_0^{\infty} \frac{\ln(x)}{x^2 + 1} dx + \int_0^{\infty} \frac{\ln\left(\frac{1}{x}\right)}{x^2 \left(\frac{1}{x^2} + 1\right)} dx = \\ &= \int_0^{\infty} \frac{\ln(x)}{x^2 + 1} dx + \int_0^{\infty} \frac{-\ln(x)}{x^2 + 1} dx = 0 \\ A + B &= \pi \end{aligned}$$