

ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\int_0^1 \frac{x^{3i} - x^{-\frac{i}{3}}}{\ln(x)} dx$$

Proposed by Henry Yang-Bangladesh

Solution by Chew Seong Cheong-Malaysia

Using the following Frullani's integral:

$$\therefore \int_0^1 \frac{x^a - x^b}{\ln(x)} dx = \ln\left(\frac{a+1}{b+1}\right)$$

valid for $a, b \in \mathbb{C}$ with $\Re(a), \Re(b) > -1$ we have:

$$\begin{aligned} \int_0^1 \frac{x^{3i} - x^{-\frac{i}{3}}}{\ln(x)} dx &= \ln\left(\frac{3i+1}{-\frac{i}{3}+1}\right) = \ln\left(\frac{3(1+3i)}{3-i}\right) = \\ &= \ln\left(\frac{3(1+3i)(3+i)}{(3-i)(3+i)}\right) = \ln\left(\frac{3(10i)}{10}\right) = \ln(3) + \ln(i) = \\ &= \ln(3) + \ln\left(e^{i\frac{\pi}{2}}\right) = \ln(3) + i\frac{\pi}{2} \end{aligned}$$

ROMANIAN MATHEMATICAL MAGAZINE