ROMANIAN MATHEMATICAL MAGAZINE

$$\textit{If } f(x)=y,\ \frac{dy}{dx}+2xy=x\cdot e^{-x^2}\ ,\ y(0)=1\ \textit{prove then:}$$

$$\int_0^{\infty} x\cdot\ln(x)\cdot f(x)dx=\frac{1-3\gamma}{8}$$

$$\textit{where } \gamma \textit{ is the Euler}-\textit{Mascheroni constant}$$

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$$\frac{dy}{dx}+2xy=x\cdot e^{-x^2}\quad \textit{IF method}\rightarrow\ \mu(x)=e^{\int 2xdx}=e^{x^2}$$

$$\mu(x)\cdot\left(\frac{dy}{dx}+2xy\right)=x\cdot e^{-x^2}\mu(x)\rightarrow \underbrace{\mu(x)\left(\frac{dy}{dx}+2xy\right)}_{\frac{d}{dx}(\mu(x)y)}=x$$

$$\frac{d}{dx}\left(e^{x^2}y\right)=x\rightarrow\ e^{x^2}y=\int xdx=\frac{x^2}{2}+C\rightarrow\ y=\frac{x^2}{2}e^{-x^2}+C\cdot e^{-x^2}$$

$$y(0)=1\rightarrow\ y(0)=C=1\quad C=1$$

$$y=f(x)=\frac{x^2}{2}e^{-x^2}+e^{-x^2}$$

$$I=\int_0^{\infty} x\ln(x)\cdot f(x)dx=\int_0^{\infty}\left(\frac{x^3}{2}e^{-x^2}+xe^{-x^2}\right)\ln(x)\,dx\ \overset{x^2=t}{\cong}$$

$$=\frac{1}{8}\int_0^{\infty} t\cdot e^{-t}\ln(t)\,dt+\frac{1}{4}\int_0^{\infty} e^{-t}\ln(t)\,dt=\frac{I_1}{8}+\frac{I_2}{4}$$

$$\textit{Note: Gamma function integral}\quad \int_0^{\infty} x^{n-1}e^{-x}dx=\Gamma(n)$$

$$\int_0^{\infty} x^{n-1}e^{-x}\ln(x)\,dx=\lim_{a\to n}\frac{\partial}{\partial a}\int_0^{\infty} x^{a-1}e^{-x}dx=\Gamma'(n)$$

$$I_1=\int_0^{\infty} t\cdot e^{-t}\ln(t)\,dt=\Gamma'(2)=1-\gamma,\qquad I_2=\int_0^{\infty} e^{-t}\ln(t)\,dt=\Gamma'(1)=-\gamma$$

Note: $\Gamma(n+1)=n\cdot\Gamma(n),\qquad \Gamma'(1)=-\gamma,\qquad \Gamma'(2)=1+\Gamma'(1)=1-\gamma$

$$I=\frac{1}{8}I_1+\frac{1}{4}I_2=\frac{1}{8}(1-\gamma)-\frac{1}{4}\gamma=\frac{1}{8}-\frac{3\gamma}{4}=\frac{1-3\gamma}{8}$$

$$Therefore\ \int_0^{\infty} x\cdot\ln(x)\cdot f(x)dx=\frac{1-3\gamma}{8}$$