ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\int_0^{\infty} \ln\left(\frac{x^2+2026^2}{x^2+2023^2}\right) dx = 23\pi$$

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$$I(a,b) = \int_0^{\infty} \ln\left(\frac{x^2+a^2}{x^2+b^2}\right) dx$$

$$I(b,b) = \int_0^{\infty} \ln\left(\frac{x^2+b^2}{x^2+b^2}\right) dx = \int_0^{\infty} \ln(1) dx = 0$$

$$\frac{dI}{da} = \int_0^{\infty} \frac{d}{da}\left(\ln(x^2+a^2) - \ln(x^2+b^2)\right) dx$$

$$\frac{dI}{da} = \int_0^{\infty} \frac{2a}{x^2+a^2} dx$$

$$\frac{dI}{da} = 2a \cdot \left[\frac{1}{a}\arctan\left(\frac{x}{a}\right)\right]_0^{\infty}$$

$$\frac{dI}{da} = 2\left(\lim_{x\to\infty} \arctan\left(\frac{x}{a}\right) - \arctan(0)\right)$$

$$\frac{dI}{da} = 2\left(\frac{\pi}{2} - 0\right) = \pi$$

$$I(a,b) = \int \pi \, da = \pi a + C(b)$$

$$I(b,b) = \pi b + C(b) = 0 \implies C(b) = -\pi b$$

$$I(a,b) = \pi(a-b)$$

$$a = 2026, \quad b = 2003$$

$$\int_0^{\infty} \ln\left(\frac{x^2+2026^2}{x^2+2003^2}\right) dx = \pi(2026-2003) = 23\pi$$