ROMANIAN MATHEMATICAL MAGAZINE

Prove that :

$$\Omega=\int_0^1\frac{\ln\left(ln\left(\frac{1}{x}\right)\right)}{\sqrt{ln\left(\frac{1}{x}\right)}}dx=-(\gamma+2\ln(2))\sqrt{\pi}$$

Proposed by Cosghun Memmedov-Azerbaijan

Solution by Shirvan Tahirov-Azerbaijan

$$\Omega=\int_0^1\frac{\ln\left(ln\left(\frac{1}{x}\right)\right)}{\sqrt{ln\left(\frac{1}{x}\right)}}dx\ \begin{cases}\ln\left(\frac{1}{x}\right)=y\\ dx=-e^{-y}dy\end{cases}$$

$$\Omega=\int_\infty^0\frac{\ln(y)}{\sqrt{y}}.(-e^{-y}dy)=\int_0^\infty e^{-y}\frac{\ln(y)}{\sqrt{y}}dy=\int_0^\infty e^{-y}.y^{-\frac{1}{2}}\ln(y)\,dy=$$

$$=\Gamma'\left(\frac{1}{2}\right)=\Gamma\left(\frac{1}{2}\right).\psi\left(\frac{1}{2}\right)=\sqrt{\pi}.(-\gamma-2\ln(2))=-(\gamma+2\ln(2))\sqrt{\pi}$$

Note :

$$\therefore\ \Gamma(z)=\int_0^\infty t^{z-1}.e^{-t}dt,\qquad \Gamma'(z)=\int_0^\infty t^{z-1}.e^{-t}\ln(t)dt$$

$$\therefore\ \psi(z)=\frac{d}{dz}ln\Gamma(z)=\frac{\Gamma'(z)}{\Gamma(z)},\qquad \Gamma'(z)=\Gamma(z).\psi(z)$$

$$\therefore\ \psi\left(k+\frac{1}{2}\right)=-\gamma-2\ln(2)+\sum_{n=1}^{k}\frac{2}{2k-1}$$

$$\therefore\ \psi\left(\frac{1}{2}\right)=-\gamma-2\ln(2),\qquad \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$$