ROMANIAN MATHEMATICAL MAGAZINE

Prove that:

$$\int_0^1 \frac{\frac{\ln(x)}{(1-x)^2}}{\frac{(1+x)^2}{\ln(x)}}\,dx = \frac{\pi^2+7\zeta(3)}{8}$$

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$$\int_0^1 \frac{\frac{\ln(x)}{(1-x)^2}}{\frac{(1+x)^2}{\ln(x)}}\,dx = \int_0^1 \left(\frac{\ln(x)}{(1-x)^2} \div \frac{(1+x)^2}{\ln(x)}\right) dx = \int_0^1 \frac{ln^2(x)}{(1-x)^2(1+x)^2}\,dx =$$

$$= \int_0^1 \frac{ln^2(x)}{((1-x)^2)^2}\,dx = \sum_{j=0}^{\infty}(j+1)\int_0^1 x^2\, ln^2(x)dx = \sum_{j=0}^{\infty}\left((j+1)\times\frac{2}{(2j+1)^3}\right) =$$

$$= 2\sum_{j=0}^{\infty}\frac{j+1}{(2j+1)^3} = \sum_{j=0}^{\infty}\left(\frac{1}{(2j+1)^2}+\frac{1}{(2j+1)^3}\right) = \sum_{j=0}^{\infty}\frac{1}{(2j+1)^2}+\sum_{j=0}^{\infty}\frac{1}{(2j+1)^3} =$$

$$= \left(1-\frac{1}{4}\right)\zeta(2)+\left(1-\frac{1}{8}\right)\zeta(3) =$$

$$= \frac{3}{4}\zeta(2)+\frac{7}{8}\zeta(3) = \frac{6\zeta(2)+7\zeta(3)}{8} = \frac{6\times\frac{\pi^2}{6}+7\zeta(3)}{8} = \frac{\pi^2+7\zeta(3)}{8}$$

$$\therefore \sum_{j=0}^{\infty}\frac{1}{(2j+1)^t} = (1-2^{-t})\zeta(t) \quad \therefore \int_0^1 x^m ln^2(x)dx = \frac{2}{(m+1)^3}$$