ROMANIAN MATHEMATICAL MAGAZINE

Find:

$$\int_0^{\frac{\pi}{4}} \operatorname{arctanh}\left(\frac{cosec(x)-\sec(x)}{cosec(x)+\sec(x)}\right)dx$$

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Solution by Shirvan Tahirov-Azerbaijan

$$I=\int_0^{\frac{\pi}{4}} \operatorname{arctanh}\left(\frac{cosec(x)-\sec(x)}{cosec(x)+\sec(x)}\right)dx=$$

$$=\int_0^{\frac{\pi}{4}} \frac{1}{2} ln\left(\frac{1+\frac{cosec(x)-\sec(x)}{cosec(x)+\sec(x)}}{1-\frac{cosec(x)-\sec(x)}{cosec(x)+\sec(x)}}\right)dx=$$

$$=\int_0^{\frac{\pi}{4}} \frac{1}{2} ln\left(\frac{cosec(x)}{\sec(x)}\right)dx=\int_0^{\frac{\pi}{4}} \frac{1}{2} ln\left(\frac{\cos(x)}{\sin(x)}\right)dx=$$

$$=\frac{1}{2}\left(\int_0^{\frac{\pi}{4}} ln(\cos(x))\,dx-\int_0^{\frac{\pi}{4}} ln(\sin(x))\,dx\right)=\frac{1}{2}(\Omega_1-\Omega_2)$$

$$\Omega_1=\int_0^{\frac{\pi}{4}} ln(\cos(x))\,dx=$$

$$\int_0^{\frac{\pi}{4}}\left(-\ln(2)-\sum_{k=0}^{\infty}\frac{(-1)^{k+1}\cos(2kx)}{k}\right)dx=$$

$$-\frac{1}{4}\pi\ln(2)+\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{k}\int_0^{\frac{\pi}{4}}\cos(2kx)\,dx=\frac{1}{2}G-\frac{1}{4}\pi\ln(2)$$

$$\Omega_2=\int_0^{\frac{\pi}{4}} ln(\sin(x))\,dx=$$

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$$\int_0^{\frac{\pi}{4}}\left(-\ln(2)-\sum_{n=1}^{\infty}\frac{\cos(2nx)}{n}\right)dx=-\frac{1}{4}\pi\ln(2)-$$

$$\sum_{n=1}^{\infty}\int_0^{\frac{\pi}{4}}\frac{\cos(2nx)}{n}dx=-\frac{1}{4}\pi\ln(2)-\sum_{n=1}^{\infty}\left[\frac{\sin(2nx)}{2n^2}\right]\Bigg|_0^{\frac{\pi}{4}}=$$

$$-\frac{1}{4}\pi\ln(2)-\sum_{n=1}^{\infty}\frac{\sin\left(\frac{\pi n}{2}\right)}{2n^2}=-\frac{1}{4}\pi\ln(2)-\frac{1}{2}G$$

$$I=\frac{1}{2}(\Omega_1-\Omega_2)=$$

$$\frac{1}{2}\left(\frac{1}{2}G-\frac{1}{4}\pi\ln(2)+\frac{1}{4}\pi\ln(2)+\frac{1}{2}G\right)=\frac{G}{2}$$