ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{\left(\frac{\sin B+\sin C}{\sin A+\sin B}\right)^2+\left(\frac{\sin C+\sin A}{\sin A+\sin B}\right)^2}{\frac{\sin^2 C}{\sin^2 A}(\sin B+\sin C)^2+\frac{\sin^2 B}{\sin^2 A}(\sin C+\sin A)^2}\ge 1$$

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$$\forall\, A',B',C',x',y',z'>0,$$

$$\frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')\underset{①}{\overset{\text{Walter Janous}}{\ge}}\sqrt{3\sum_{cyc}A'B'}$$

$$\text{Now, }\sum_{cyc}\frac{\left(\frac{\sin B+\sin C}{\sin A+\sin B}\right)^2+\left(\frac{\sin C+\sin A}{\sin A+\sin B}\right)^2}{\frac{\sin^2 C}{\sin^2 A}(\sin B+\sin C)^2+\frac{\sin^2 B}{\sin^2 A}(\sin C+\sin A)^2}$$

$$=\sum_{cyc}\frac{\sin^2 A\left(\frac{\left(\frac{\sin B+\sin C}{\sin A+\sin B}\right)^2+\left(\frac{\sin C+\sin A}{\sin A+\sin B}\right)^2}{(\sin B+\sin C)^2(\sin C+\sin A)^2}\right)}{\frac{\sin^2 C}{(\sin C+\sin A)^2}+\frac{\sin^2 B}{(\sin B+\sin C)^2}}=\sum_{cyc}\frac{\frac{\sin^2 A}{(\sin A+\sin B)^2}\cdot\left(\frac{1}{(\sin B+\sin C)^2}+\frac{1}{(\sin C+\sin A)^2}\right)}{\frac{\sin^2 B}{(\sin B+\sin C)^2}+\frac{\sin^2 C}{(\sin C+\sin A)^2}}$$

$$=\frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')$$

$$\left(\begin{array}{l}x'=\frac{\sin^2 A}{(\sin A+\sin B)^2},y'=\frac{\sin^2 B}{(\sin B+\sin C)^2},z'=\frac{\sin^2 C}{(\sin C+\sin A)^2},\\ A'=\frac{1}{(\sin A+\sin B)^2},B'=\frac{1}{(\sin B+\sin C)^2},C'=\frac{1}{(\sin C+\sin A)^2}\end{array}\right)$$

$$\overset{\text{via } ①}{\ge}\sqrt{3\sum_{cyc}\frac{1}{(\sin A+\sin B)^2(\sin B+\sin C)^2}}\overset{\text{AM-GM}}{\ge}$$

$$\frac{3}{\left(\sqrt[3]{(\sin A+\sin B)(\sin B+\sin C)(\sin C+\sin A)}\right)^2}\overset{\text{AM-GM}}{\ge}\frac{3}{\left(\frac{2(\sin A+\sin B+\sin C)}{3}\right)^2}\overset{\text{Jensen}}{\ge}$$

$$\frac{3}{(\sqrt{3})^2}=1\text{ and so, }\sum_{cyc}\frac{\left(\frac{\sin B+\sin C}{\sin A+\sin B}\right)^2+\left(\frac{\sin C+\sin A}{\sin A+\sin B}\right)^2}{\frac{\sin^2 C}{\sin^2 A}(\sin B+\sin C)^2+\frac{\sin^2 B}{\sin^2 A}(\sin C+\sin A)^2}\ge 1\ \forall\,\Delta\text{ ABC},$$

'' $=$ '' iff Δ ABC is equilateral (QED)