ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{\csc^4\frac{B}{2}\csc^4\frac{C}{2}\left(\csc^6\frac{B}{2}+\csc^6\frac{C}{2}\right)}{\csc^5\frac{B}{2}+\csc^5\frac{C}{2}}\geq 3\cdot 8^3$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\text{LHS}\overset{\text{Chebyshev}}{\geq}\sum_{cyc}\frac{\csc^4\frac{B}{2}\csc^4\frac{C}{2}\left(\csc^5\frac{B}{2}+\csc^5\frac{C}{2}\right)\left(\csc\frac{B}{2}+\csc\frac{C}{2}\right)}{2\left(\csc^5\frac{B}{2}+\csc^5\frac{C}{2}\right)}\overset{\text{AM-GM}}{\geq}$$

$$\frac{3}{2}\cdot\sqrt[3]{\left(\prod_{cyc}\csc^8\frac{A}{2}\right)\cdot\prod_{cyc}\left(\csc\frac{B}{2}+\csc\frac{C}{2}\right)}\overset{\text{Cesaro}}{\geq}\frac{3}{2}\cdot\sqrt[3]{\left(\prod_{cyc}\csc^8\frac{A}{2}\right).8\prod_{cyc}\csc\frac{A}{2}}$$

$$=3.\prod_{cyc}\csc^3\frac{A}{2}=3.\left(\frac{4R}{r}\right)^3\overset{\text{Euler}}{\geq}3.8^3\therefore\sum_{cyc}\frac{\csc^4\frac{B}{2}\csc^4\frac{C}{2}\left(\csc^6\frac{B}{2}+\csc^6\frac{C}{2}\right)}{\csc^5\frac{B}{2}+\csc^5\frac{C}{2}}\geq 3.8^3$$

$\forall\,\Delta$ ABC, "=" iff Δ ABC is equilateral (QED)