ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{r_a^{n-1}(r_c^{n+1}+r_a^{n+1})}{r_b^n r_c(r_a^n r_b^n+r_c^{2n})}+\frac{r_b^{n-1}(r_a^{n+1}+r_b^{n+1})}{r_c^n r_a(r_c^n r_b^n+r_a^{2n})}+\frac{r_c^{n-1}(r_c^{n+1}+r_b^{n+1})}{r_a^n r_b(r_a^n r_c^n+r_b^{2n})}\ge\frac{2}{R}\cdot\left(\frac{2}{3R}\right)^n, n\in\mathbb{N}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc}\frac{r_a^{n-1}(r_c^{n+1}+r_a^{n+1})}{r_b^n r_c(r_a^n r_b^n+r_c^{2n})}=\sum_{cyc}\frac{\left(\frac{r_a}{r_b}\right)^n}{\left(\left(\frac{r_b}{r_c}\right)^n+\left(\frac{r_c}{r_a}\right)^n\right)}\cdot\frac{1}{(r_a r_c)^{n+1}}(r_c^{n+1}+r_a^{n+1})=$$

$$=\sum_{cyc}\frac{\left(\frac{r_a}{r_b}\right)^n}{\left(\frac{r_b}{r_c}\right)^n+\left(\frac{r_c}{r_a}\right)^n}\left(\frac{1}{r_a^{n+1}}+\frac{1}{r_c^{n+1}}\right)$$

The left side of the inequality satisfies the conditions of Walter Janous' theorem:

x, y, z, a', b', c' – *positive real numbers*:

$$\frac{x}{y+z}(b'+c')+\frac{y}{x+z}(c'+a')+\frac{z}{x+y}(a'+b')\ge\sqrt{3\sum_{cyc}a'b'}$$

Here: $x=\left(\frac{r_c}{r_a}\right)^n, y=\left(\frac{r_a}{r_b}\right)^n, z=\left(\frac{r_b}{r_c}\right)^n, a'=\frac{1}{r_a^{n+1}}, b'=\frac{1}{r_b^{n+1}}, c'=\frac{1}{r_c^{n+1}}$

$$LHS=\sqrt{3}\left(\left(\frac{1}{r_a r_b}\right)^{n+1}+\left(\frac{1}{r_c r_b}\right)^{n+1}+\left(\frac{1}{r_a r_c}\right)^{n+1}\right)^{\frac{1}{2}}\overset{AM-GM}{\ge}$$

$$\sqrt{3}.\sqrt{3}\left(\left(\frac{1}{r_a r_b r_c}\right)^{2n+2}\right)^{\frac{1}{6}}=3\left(\frac{1}{S^2 r}\right)^{\frac{n+1}{3}}\overset{Euler,Mitrinovic}{\ge}$$

$$3\left(\frac{8}{27R^3}\right)^{\frac{n+1}{3}}=\left(\frac{2}{3R}\right)^{n+1}=\frac{2}{R}\cdot\left(\frac{2}{3R}\right)^n$$

Equality holds for an equilateral triangle.