ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{\sin\frac{A}{2}\left(\sin^3\frac{C}{2}+\sin^3\frac{A}{2}\right)}{\sin^2\frac{B}{2}\sin\frac{C}{2}\left(\sin^2\frac{A}{2}\sin^2\frac{B}{2}+\sin^4\frac{C}{2}\right)}\geq 24$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\forall A', B', C', x', y', z' > 0,$$

$$\frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')\underbrace{\overset{\text{Walter Janous}}{\geq}}_{①}\sqrt{3\sum_{cyc}A'B'}$$

$$\text{Now, }\sum_{cyc}\frac{\sin\frac{A}{2}\left(\sin^3\frac{C}{2}+\sin^3\frac{A}{2}\right)}{\sin^2\frac{B}{2}\sin\frac{C}{2}\left(\sin^2\frac{A}{2}\sin^2\frac{B}{2}+\sin^4\frac{C}{2}\right)}=\sum_{cyc}\frac{\sin^2\frac{A}{2}\sin\frac{C}{2}\cdot\left(\frac{\sin^3\frac{C}{2}+\sin^3\frac{A}{2}}{\sin\frac{C}{2}\sin\frac{A}{2}}\right)}{\sin^2\frac{B}{2}\sin\frac{C}{2}\left(\sin^2\frac{A}{2}\sin^2\frac{B}{2}+\sin^4\frac{C}{2}\right)}$$

$$=\sum_{cyc}\frac{\frac{\sin^2\frac{A}{2}}{\sin^2\frac{B}{2}}\cdot\left(\frac{\sin^3\frac{C}{2}+\sin^3\frac{A}{2}}{\sin^3\frac{C}{2}\sin^3\frac{A}{2}}\right)}{\frac{\sin^2\frac{A}{2}\sin^2\frac{B}{2}+\sin^4\frac{C}{2}}{\sin^2\frac{C}{2}\sin^2\frac{A}{2}}}=\sum_{cyc}\frac{\frac{\sin^2\frac{A}{2}}{\sin^2\frac{B}{2}}\cdot\left(\frac{1}{\sin^3\frac{C}{2}}+\frac{1}{\sin^3\frac{A}{2}}\right)}{\frac{\sin^2\frac{B}{2}}{\sin^2\frac{C}{2}}+\frac{\sin^2\frac{C}{2}}{\sin^2\frac{A}{2}}}$$

$$=\frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')$$

$$\left(x'=\frac{\sin^2\frac{A}{2}}{\sin^2\frac{B}{2}},y'=\frac{\sin^2\frac{B}{2}}{\sin^2\frac{C}{2}},z'=\frac{\sin^2\frac{C}{2}}{\sin^2\frac{A}{2}},A'=\frac{1}{\sin^3\frac{B}{2}},B'=\frac{1}{\sin^3\frac{C}{2}},C'=\frac{1}{\sin^3\frac{A}{2}}\right)$$

$$\overset{\text{via }①}{\geq}\sqrt{3\sum_{cyc}\frac{1}{\sin^3\frac{B}{2}\sin^3\frac{C}{2}}}\overset{\text{AM-GM}}{\geq}3.\sqrt[6]{\frac{1}{\sin^6\frac{A}{2}\sin^6\frac{B}{2}\sin^6\frac{C}{2}}}=\frac{12R}{r}\overset{\text{Euler}}{\geq}24$$

$$\therefore\sum_{cyc}\frac{\sin\frac{A}{2}\left(\sin^3\frac{C}{2}+\sin^3\frac{A}{2}\right)}{\sin^2\frac{B}{2}\sin\frac{C}{2}\left(\sin^2\frac{A}{2}\sin^2\frac{B}{2}+\sin^4\frac{C}{2}\right)}\geq 24\ \forall\ \Delta\text{ ABC},$$

$$''='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$