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In acute ΔABC the following relationship holds:

$$a^2\sqrt{\tan A} + b^2\sqrt{\tan B} + c^2\sqrt{\tan C} \geq 36\sqrt[4]{3}r^2$$

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$$\textit{As triangle is acute then}:$$

$$\sum \tan A \overset{Jensen}{\geq} 3\tan\frac{A+B+C}{3} = 3\tan\frac{\pi}{3} = 3\sqrt{3}$$

$$\textit{We know that in any } \Delta ABC: \sum \tan A = \prod \tan A \geq 3\sqrt{3} = \left(\sqrt{3}\right)^3 \ (1)$$

$$a^2\sqrt{\tan A} + b^2\sqrt{\tan B} + c^2\sqrt{\tan C} \overset{Chebyshev}{\geq} \frac{1}{3}\left(\sum a^2\right)\left(\sum \sqrt{\tan A}\right) \overset{AM-GM}{\geq}$$

$$\geq \frac{1}{3}\left(\sum a^2\right).3.\sqrt[6]{\prod \tan A} \overset{\substack{Neuberg\\(1)}}{\geq} \frac{1}{3}36r^2.\sqrt[4]{3} = 36\sqrt[4]{3}r^2$$

Equality holds for an equilateral triangle.