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In acute $\triangle ABC$ the following relationship holds:

$$a \tan\left(\frac{A}{2}\right) \sqrt{\sin A} + b \tan\left(\frac{B}{2}\right) \sqrt{\sin B} + c \tan\left(\frac{C}{2}\right) \sqrt{\sin C} > 4r$$

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$$\sum_{cyc} a \tan\left(\frac{A}{2}\right) \sin(A) > 4r$$

Lemma 1 (The Relationship between Inradius and Tangent):

$$\tan\left(\frac{A}{2}\right) = \frac{r}{s-a}, \tan\left(\frac{B}{2}\right) = \frac{r}{s-b}, \tan\left(\frac{C}{2}\right) = \frac{r}{s-c}$$

Lemma 2 (Inequality of Sine in Acute Triangles):

$$\text{interval } \left(0; \frac{\pi}{2}\right) \quad 0 < \sin A < 1 \quad \begin{cases} \sqrt{\sin A} > \sin A \\ \sqrt{\sin B} > \sin B \\ \sqrt{\sin C} > \sin C \end{cases}$$

Lemma 3 (Convexity and Jensen's Inequality):

$$f(a) = \frac{a^2}{s-a} \text{ for } a \in (0; s)$$

Since its second derivative $f''(a) = \frac{2s^2}{(s-a)^3}$ is strictly positive, the function is strictly convex. By Jensen's Inequality, the cyclic sum $\sum_{cyc} \frac{a^2}{s-a}$ is minimized when $a = b = c$, corresponding to an equilateral triangle.

$$\text{LHS} \stackrel{\text{Lemma 1}}{\cong} \sum_{cyc} a \left(\frac{r}{s-a}\right) \sqrt{\sin A}$$

$$\stackrel{\text{Lemma 2}}{\rightarrow} r \sum_{cyc} \left(\frac{a}{s-a}\right) \sqrt{\sin A} > r \sum_{cyc} \left(\frac{a}{s-a}\right) \sin A$$

$$\text{Theorem sine } \sin A = \frac{a}{2R} \quad \sin B = \frac{b}{2R} \quad \sin C = \frac{c}{2R}$$

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$$r \sum_{cyc} \frac{a}{s-a} \sin A = \frac{r}{2R} \sum_{cyc} \frac{a^2}{s-a}$$

Lemma 3: $\left(\sum_{cyc} \frac{a}{s-a} \right)_{min} \rightarrow a = b = c \quad s = \frac{a+b+c}{2} = \frac{3a}{2}, s-a = \frac{a}{2}$

$$\sum_{cyc} \frac{a^2}{s-a} \geq 3 \frac{a^2}{s-a} = 6a \rightarrow LHS > \frac{r}{2R} 6a = \frac{3ar}{R}$$

$$a = b = c = R\sqrt{3} \quad LHS > \frac{3ar}{R} = 3r\sqrt{3}$$

$$3\sqrt{3} > 4 \rightarrow \sum_{cyc} a \tan\left(\frac{A}{2}\right) \sin(A) > 4r$$