ROMANIAN MATHEMATICAL MAGAZINE

Let Ω Brocard point of ΔABC. O_a, O_b, O_c circumcenters of $\Delta B\Omega C, \Delta A\Omega C, \Delta A\Omega B$. I, I_a, I_b, I_c – incenter and excenters of ΔABC. Prove that:

$$\sum \frac{OO_a \cdot c}{AI \cdot AI_a} \geq \frac{9r}{s}$$

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$$Lemma: AI = \frac{r}{\sin\left(\frac{A}{2}\right)},\ AI_a = \frac{r_a}{\sin\left(\frac{A}{2}\right)} \text{ and } ab + bc + ca \geq 36r^2$$

$$Let\ OO_{a\cap BC} = K,\ OK = x \text{ and } O_aK = y.\ \cot A = \frac{2x}{a} \text{ and } \cot C = \frac{2y}{a} \Rightarrow$$

$$\Rightarrow OO_a = \frac{a}{2} \cdot \frac{\sin B}{\sin A \cdot \sin C} = \frac{a}{2} \cdot \frac{\frac{b}{2R}}{\frac{a}{2R} \cdot \frac{c}{2R}} = \frac{bR}{c}$$

$$LHS = \sum \frac{\frac{bR}{c} \cdot c}{\frac{rr_a}{\sin^2\left(\frac{A}{2}\right)}} = \sum \frac{bR}{\frac{rr_a}{\sin^2\left(\frac{A}{2}\right)}} = \frac{R}{r} \sum \frac{b \cdot \sin^2\left(\frac{A}{2}\right)}{r_a} = \frac{R}{r} \sum \frac{b \cdot \sin^2\left(\frac{A}{2}\right)}{s \cdot \tan\left(\frac{A}{2}\right)} =$$

$$= \frac{R}{r} \sum \frac{b \sin A}{2s} = \sum \frac{ab}{4rs} \geq \frac{36r^2}{4rs} = \frac{9r}{s}$$

Equality holds for $a = b = c$.