ROMANIAN MATHEMATICAL MAGAZINE

$H,\ O$ *orhocenter and circumcenter of acute* $\Delta ABC$.
$O_a,\ O_b,\ O_c$ *circumcenters of* $\Delta BHC,\ \Delta AHC,\ \Delta AHB$.
$I_a,\ I_b,\ I_c$ − *excenters of* $\Delta ABC$. *Prove that*:

$$\sum \frac{OO_a}{AI \cdot AI_a} \le \frac{1}{3}\left(\frac{1}{r}+\frac{1}{R}\right)$$

*Proposed by Sarkhan Adgozalov-Georgia*

*Solution by Qurban Muellim-Azerbaijan*

$$Lemma:\ AI = \frac{r}{\sin\left(\frac{A}{2}\right)}\ and\ AI_a = \frac{r_a}{\sin\left(\frac{A}{2}\right)}$$

$$Let\ OO_a = 2x,\ \cot A = \frac{2x}{a} \Rightarrow\ 2x = a.\cot A\ \Rightarrow\ OO_a = 2R.\cos A$$

$$LHS = \sum \frac{OO_a}{AI \cdot AI_a} = \sum \frac{2R\cos A}{\frac{rr_a}{\sin^2\left(\frac{A}{2}\right)}} = \sum \frac{2R\cos A \sin^2\left(\frac{A}{2}\right)}{rs.\tan\left(\frac{A}{2}\right)} = \sum \frac{R\cos A \sin A}{rs} =$$

$$= \sum \frac{R \cdot \sin(2A)}{2rs} = \frac{R}{2rs}\sum \sin(2A) = \frac{R}{2rs}\prod \sin(A) = \frac{4R}{2rs} \cdot \frac{abc}{8R^3} = \frac{1}{R}$$

$$LHS = \frac{1}{R} = \frac{1}{3}\left(\frac{2}{R}+\frac{1}{R}\right) \le \frac{1}{3}\left(\frac{2}{2r}+\frac{1}{R}\right) = \frac{1}{3}\left(\frac{1}{r}+\frac{1}{R}\right)$$

*Equality holds for* $a = b = c$.