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Let ΔDEF be the Gergonne's triangle of ΔABC.
I_a, I_b, I_c – excenters of ΔABC. Prove that:

$$9\sqrt{3}Rr \geq \sum EF\,.\,AI_a \geq 18\sqrt{3}r^2$$

Proposed by Sarkhan Adgozalov-Georgia

Solution by Qurban Muellim-Azerbaijan

$$AI_a = \frac{r_a}{\sin\left(\frac{A}{2}\right)}$$

$$\frac{FE}{\sin\left(\measuredangle FDE = 90 - \frac{A}{2}\right)} = 2r \Rightarrow FE = 2r\cos\left(\frac{A}{2}\right)$$

$$EF.AI_a = 2r\cos\left(\frac{A}{2}\right)\cdot\frac{r_a}{\sin\left(\frac{A}{2}\right)} = 2rr_a\cdot\cot\left(\frac{A}{2}\right) = 2rr_a\cdot\frac{s}{r_a} = 2rs$$

$$\sum EF\cdot AI_a = \sum 2rs = 6rs$$

$$6rs \geq 6r\cdot 3\sqrt{3}r = 18\sqrt{3}r^2\ (1)$$

$$6rs \leq 6r\cdot\frac{3\sqrt{3}R}{2} = 9\sqrt{3}Rr\ \ (2)$$

$$(1)\wedge(2) \Rightarrow 9\sqrt{3}Rr \geq \sum EF\cdot AI_a \geq 18\sqrt{3}r^2$$

Equality holds for $a = b = c$.